

Bifurcation curves of a logistic equation when the linear growth rate crosses a second eigenvalue*

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Abstract

We construct the global bifurcation curves, solutions versus level of harvesting, for the steady states of a diffusive logistic equation on a bounded domain, under Dirichlet boundary conditions and other appropriate hypotheses, when a , the linear growth rate of the population, is below $\lambda_2 + \delta$. Here λ_2 is the second eigenvalue of the Dirichlet Laplacian on the domain and $\delta > 0$. Such curves have been obtained before, but only for a in a right neighborhood of the first eigenvalue. Our analysis provides the exact number of solutions of the equation for $a \leq \lambda_2$ and new information on the number of solutions for $a > \lambda_2$.

1 Introduction

Diffusive logistic equations with harvesting are equations of the form

$$-\Delta u = au - f(u) - ch \quad (1)$$

in a domain Ω , with some boundary conditions, here taken to be homogeneous Dirichlet, for a competition term f and a harvesting function h and level c . In [8], [12] and the references therein the reader may find recent work regarding this subject.

In [13] the authors obtained global bifurcation diagrams for solutions of (1) when the competition term is proportional to the square of the population u . The original motivation for our study was somewhat limited. We wanted to be able to deal with other competition terms whose second derivative vanishes at zero. Building on the work in [13], we achieve our original goal and, more importantly, prove several new results. We denote by λ_1

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and λ_2 the first and second eigenvalues of the Dirichlet Laplacian on Ω , respectively. Whereas in [13] the authors, who were seeking positive solutions, obtained global bifurcation curves for the parameter a below $\lambda_1 + \delta$, under suitable additional hypotheses we obtain global bifurcation curves for any a below $\lambda_2 + \delta$. Part of the solutions on these curves will change sign.

We should mention that this paper is also related to problems addressing the so called jumping nonlinearities. Here, unlike the most common hypothesis (see, for example, [7]), the nonlinearity is not asymptotically linear on one of the ends of the real line.

Let us now state our most important results. Let Ω be a smooth bounded domain in \mathbb{R}^N with $N \geq 1$, $p > N$ and $\mathcal{H} = \{u \in W^{2,p}(\Omega) : u = 0 \text{ on } \partial\Omega\}$. We are interested in weak solutions of the equation (1) belonging to the space \mathcal{H} . In the first sections of the paper we just assume

- (i) $f \in C^2(\mathbb{R})$.
- (ii) $f(u) = 0$ for $u \leq M$, and $f(u) > 0$ for $u > M$; throughout $M \geq 0$ is fixed.
- (iii) $f''(u) \geq 0$.
- (iv) $\lim_{u \rightarrow +\infty} \frac{f(u)}{u} = +\infty$.
- (a) $h \in L^\infty(\Omega)$.
- (b) $h \geq 0$ in Ω and $h > 0$ on a set of positive measure.

For a below λ_2 we prove

Theorem 1.1. *Suppose f satisfies (i)-(iv) and h satisfies (a)-(b). Fix $\lambda_1 < a < \lambda_2$. The set of solutions (c, u) of (1) is a connected one dimensional manifold \mathcal{M} of class C^1 in $\mathbb{R} \times \mathcal{H}$. We have*

$$\mathcal{M} = \mathcal{M}^\sharp \cup \{\mathbf{p}_*\} \cup \mathcal{M}^*,$$

where $\{\mathbf{p}_*\}$ connects \mathcal{M}^\sharp and \mathcal{M}^* . Here

- \mathcal{M}^\sharp is the manifold of nondegenerate solutions with Morse index equal to one, which is a graph $\{(c, u^\sharp(c)) : c \in]-\infty, c_*[\}$.
- $\mathbf{p}_* = (c_*, u_*)$ is a degenerate solution with Morse index equal to zero.
- \mathcal{M}^* is the manifold of stable solutions, which is a graph $\{(c, u^*(c)) : c \in]-\infty, c_*[\}$.

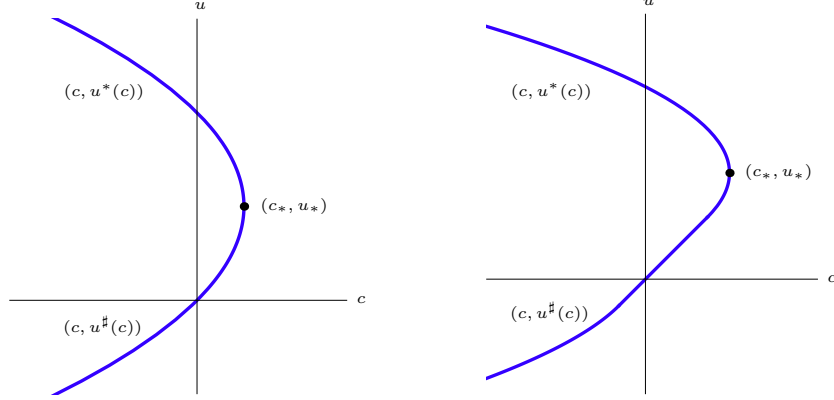


Figure 1: Bifurcation curve for $\lambda_1 < a < \lambda_2$. On the left $M = 0$ and on the right $M > 0$.

Theorem 1.1 is illustrated in Figure 1.

To study (1) for $a \geq \lambda_2$ we make additional assumptions. Specifically, we assume

(α) λ_2 is simple, with eigenspace spanned by ψ .

(c) $\int h\psi \neq 0$.

When the region of integration is omitted it is understood to be Ω . We denote by ϕ the first eigenfunction of the Dirichlet Laplacian satisfying $\max_{\Omega} \phi = 1$, and we also normalize the second eigenfunction ψ so $\max_{\Omega} \psi = 1$. We define

$$\beta = -\min_{\Omega} \psi, \quad (2)$$

so that $\beta > 0$. For a equal to λ_2 we prove

Theorem 1.2. *Suppose f satisfies (i)-(iv), (α) holds and h satisfies (a)-(c). Fix $a = \lambda_2$. The set of solutions (c, u) of (1) is a connected one dimensional manifold \mathcal{M} of class C^1 in $\mathbb{R} \times \mathcal{H}$. We have*

$$\mathcal{M} = \mathcal{M}^b \cup \mathcal{L} \cup \mathcal{M}^\sharp \cup \{\mathbf{p}_*\} \cup \mathcal{M}^*,$$

where \mathcal{L} connects \mathcal{M}^b and \mathcal{M}^\sharp , and $\{\mathbf{p}_*\}$ connects \mathcal{M}^\sharp and \mathcal{M}^* . Here

- \mathcal{M}^b is a manifold of nondegenerate solutions with Morse index equal to one, which is a graph $\{(c, u^b(c)) : c \in]-\infty, 0[\}$.
- \mathcal{L} is a segment (a point in the case $M = 0$) of degenerate solutions with Morse index equal to one, $\{(0, t\psi) : t \in [-\frac{M}{\beta}, M] \}$.

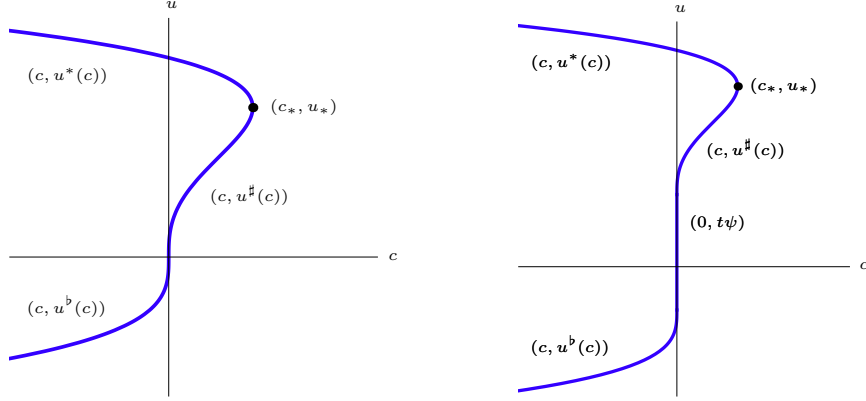


Figure 2: Bifurcation curve for $a = \lambda_2$. On the left $M = 0$ and on the right $M > 0$.

- $\mathcal{M}^\#$ is a manifold of nondegenerate solutions with Morse index equal to one, which is a graph $\{(c, u^\#(c)) : c \in]0, c_*[\}$.
- $\mathbf{p}_* = (c_*, u_*)$ is a degenerate solution with Morse index equal to zero.
- \mathcal{M}^* is the manifold of stable solutions, which is a graph $\{(c, u^*(c)) : c \in]-\infty, c_*[\}$.

Theorem 1.2 is illustrated in Figure 2.

Remark 1.3. If $M = 0$ the set of solutions with Morse index equal to one is the graph of a continuous function

$$u^{\#b}(c) = \begin{cases} u^b(c) & \text{for } c < 0, \\ 0 & \text{for } c = 0, \\ u^\#(c) & \text{for } 0 < c < c_*, \end{cases}$$

which is not differentiable at zero.

We can go a little bit beyond λ_2 provided we strengthen **(b)** to **(b)'** $h > 0$ a.e. in Ω .

To fix ideas, without loss of generality, suppose

$$\int h\psi < 0. \tag{3}$$

We define

$$\mathcal{S} := \{y \in \mathcal{H} : \int y\psi = 0\}. \tag{4}$$

For a above λ_2 we prove

Theorem 1.4. Suppose f satisfies (i)-(iv), (α) holds and h satisfies (a), (b)', (c). Without loss of generality, suppose (3) is true. There exists $\delta > 0$ such that the following holds. Fix $\lambda_2 < a < \lambda_2 + \delta$. The set of solutions (c, u) of (1) is a connected one dimensional manifold \mathcal{M} of class C^1 in $\mathbb{R} \times \mathcal{H}$. We have \mathcal{M} is the disjoint union

$$\mathcal{M} = \mathcal{M}^b \cup \{\mathbf{p}_b\} \cup \mathcal{M}^\natural \cup \{\mathbf{p}_\sharp\} \cup \mathcal{M}^\sharp \cup \{\mathbf{p}_*\} \cup \mathcal{M}^*,$$

where $\{\mathbf{p}_b\}$ connects \mathcal{M}^b and \mathcal{M}^\natural , $\{\mathbf{p}_\sharp\}$ connects \mathcal{M}^\natural and \mathcal{M}^\sharp , and $\{\mathbf{p}_*\}$ connects \mathcal{M}^\sharp and \mathcal{M}^* . Here

- \mathcal{M}^b is a manifold of nondegenerate solutions with Morse index equal to one, which is a graph $\{(c, u^b(c)) : c \in]-\infty, c_b]\}$.
- $\mathbf{p}_b = (c_b, u_b)$ is a degenerate solution with Morse index equal to one.
- \mathcal{M}^\natural is a manifold of solutions with Morse index equal to one or to two,

$$\{(c^\natural(t), u^\natural(t)) : u^\natural(t) = t\psi + y^\natural(t), t \in J\},$$

with $c^\natural : J \rightarrow \mathbb{R}$, $y^\natural : J \rightarrow \mathcal{S}$ and $J =]-\frac{M}{\beta} - \varepsilon_b, M + \varepsilon_\sharp[$, for some $\varepsilon_b, \varepsilon_\sharp > 0$.

- $\mathbf{p}_\sharp = (c_\sharp, u_\sharp)$ is a degenerate solution with Morse index equal to one.
- \mathcal{M}^\sharp is a manifold of nondegenerate solutions with Morse index equal to one, which is a graph $\{(c, u^\sharp(c)) : c \in]c_\sharp, c_*]\}$.
- $\mathbf{p}_* = (c_*, u_*)$ is a degenerate solution with Morse index equal to zero.
- \mathcal{M}^* is the manifold of stable solutions, which is a graph $\{(c, u^*(c)) : c \in]-\infty, c_*]\}$.

We have $(c^\natural)'(0) < 0$ and

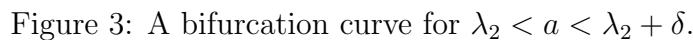
$$\lim_{t \rightarrow -\frac{M}{\beta} - \varepsilon_b} (c^\natural(t), u^\natural(t)) = (c_b, u_b), \quad \lim_{t \rightarrow M + \varepsilon_\sharp} (c^\natural(t), u^\natural(t)) = (c_\sharp, u_\sharp).$$

In particular, if $|c|$ is sufficiently small, then (1) has at least four solutions.

Theorem 1.4 is illustrated in Figure 3.

Remark 1.5. The sign of $(c^\natural)'$ allows us to classify the solutions on \mathcal{M}^\natural . Assuming (3) holds

- If $(c^\natural)'(t) > 0$, then $(c^\natural(t), u^\natural(t))$ is a nondegenerate solution with Morse index equal to one.



- The above three theorems are our main results. In addition, we also prove Theorem 3.1 (degenerate solutions with Morse index equal to zero), Proposition 3.4 (behavior of c along the curve of degenerate solutions with Morse index equal to zero), Proposition 4.2 (stable solutions are superharmonic for small $|c|$), Theorem 4.6 (existence of at least three solutions for $a > \lambda_2$, a not an eigenvalue and small $|c|$) and Proposition 7.1 (solutions for the case $c = 0$, around and bifurcating from $(\lambda_2, 0)$).

Our main tools are bifurcation theory, the Morse indices, critical points at infinity and the methods of elliptic equations. With regard to the first, the simplest approach for our purposes is to make the best choice of coordinates in each circumstance and then apply the Implicit Function Theorem. This also allows us to simplify some of the original arguments (namely in what concerns [13, Lemma 4.3]).

The organization of this paper is as follows. In Section 2 we consider stable solutions of the equation with no harvesting. In Section 3 we consider degenerate solutions with Morse index equal to zero. In Section 4 we consider stable solutions, solutions around a degenerate solution with Morse index equal to zero, solutions around zero, and mountain pass solutions. In Section 5 we discuss global bifurcation below λ_2 and prove Theorem 1.1. In Section 6 we discuss global bifurcation at λ_2 and prove Theorem 1.2. In Section 7 we discuss bifurcation in a right neighborhood of λ_2 and start the proof of Theorem 1.4. Finally, in order to extend the curves obtained in Section 7 for all negative values of c , and complete the proof of Theorem 1.4, we need a somewhat delicate argument to prove there are no degenerate solutions with Morse index equal to one at infinity for $a < \lambda_2 + \delta$. In Section 8 we carry it out.

We finish the Introduction with a word of caution about our terminology. For simplicity, we sometimes refer to a solution $(a, u, c) \in \mathbb{R} \times \mathcal{H} \times \mathbb{R}$ of (1) in an abbreviated manner, by (c, u) when a is fixed, or simply by u when both a and c are fixed. And we may also refer to some property of a solution (a, u, c) , like positivity, meaning the second component u has that property.

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2 Stable solutions of the equation with no harvesting

Throughout this section we assume $c = 0$, so we consider the equation

$$-\Delta u = au - f(u). \quad (5)$$

Here the main result is

Theorem 2.1 (\mathcal{C}_\dagger , solutions of (5) bifurcating from $(\lambda_1, 0)$). *Suppose f satisfies (i)-(iv). The set of positive solutions (a, u) of (5) is a connected one dimensional manifold \mathcal{C}_\dagger of class C^1 in $\mathbb{R} \times \mathcal{H}$. The manifold is the union of the segment $\{(\lambda_1, t\phi) : t \in]0, M]\}$ with a graph $\{(a, u_\dagger(a)) : a \in]\lambda_1, +\infty[\}$. The solutions are strictly increasing along \mathcal{C}_\dagger . For $a > \lambda_1$ every positive solution is stable and, at each a , equation (5) has no other stable solution besides $u_\dagger(a)$.*

Remark 2.2. In Theorem 2.1 we may relax assumption (iii) to

(iii)' $u \mapsto \frac{f(u)}{u}$ is increasing (not necessarily strictly).

We define

$$\mathcal{R} := \{y \in \mathcal{H} : \int y \phi = 0\}.$$

Lemma 2.3 (Initial portion of \mathcal{C}_\dagger). *Suppose $M > 0$. There exists $\delta > 0$ and C^1 functions $a_\dagger : J \rightarrow \mathbb{R}$ and $y_\dagger : J \rightarrow \mathcal{R}$, where $J =]-\infty, M + \delta[$, such that the map $t \mapsto (a_\dagger(t), t\phi + y_\dagger(t))$, defined in J , with $a_\dagger(t) = \lambda_1$ and $y_\dagger(t) = 0$ for $t \in]-\infty, M]$, parametrizes a curve \mathcal{C}_\dagger of solutions of (5). There exists a neighborhood of $\mathcal{C}_\dagger \setminus \{(\lambda_1, 0)\}$ in $\mathbb{R} \times \mathcal{H}$ such that the solutions of (5) in this neighborhood lie on \mathcal{C}_\dagger .*

Sketch of the proof. [‡] Apply the Implicit Function Theorem to the function $g : \mathbb{R}^2 \times \mathcal{R} \rightarrow L^p(\Omega)$, defined by

$$g(a, t, y) = \Delta(t\phi + y) + a(t\phi + y) - f(t\phi + y)$$

at $(\lambda_1, t_0, 0)$ with $t_0 \in]-\infty, 0[\cup]0, M]$. □

Of course the line $\mathcal{A} \subset \mathbb{R} \times \mathcal{H}$ parametrized by $a \mapsto (a, 0)$ is also a curve of solutions of (5). From the classical paper [9, Theorem 1.7] we know that $(\lambda_1, 0)$ is a bifurcation point. We deduce that the statement of Lemma 2.3 also holds when $M = 0$. In both cases, $M > 0$ and $M = 0$, in a neighborhood of $(\lambda_1, 0)$ the solutions of (5) lie on $\mathcal{A} \cup \mathcal{C}_\dagger$.

Let δ be as in Lemma 2.3 and $u_\dagger(t) = t\phi + y_\dagger(t)$. Reducing δ if necessary, we may assume $u_\dagger(t) > 0$ and $\max_\Omega u_\dagger(t) > M$ for $t \in]M, M + \delta[$. Indeed,

$$u_\dagger(t) = M\phi + (t - M)\phi + o(|t - M|), \quad (6)$$

as $y'_\dagger(M) = 0$, and ϕ has negative exterior normal derivative on $\partial\Omega$. It follows

$$a_\dagger(t) > \lambda_1 \quad \text{if } t \in]M, M + \delta[.$$

To see this, we multiply both sides of (5) by ϕ and integrate over Ω ,

$$(a - \lambda_1) \int u \phi = \int f(u) \phi. \quad (7)$$

Notice $f(u_\dagger(t)) \not\equiv 0$ because $\max_\Omega u_\dagger(t) > M$.

Let u be a solution of (5), μ be the smallest eigenvalue of the linearized problem at u and $v \in \mathcal{H}$ be a corresponding eigenfunction. So

$$-(\Delta v + av - f'(u)v) = \mu v. \quad (8)$$

[‡]For the full proof see the Appendix 9.

We recall that either v or $-v$ is strictly positive everywhere in Ω . The solution u is said to be stable if the smallest eigenvalue of the linearized problem is positive. The Morse index of the solution u is the number of negative eigenvalues of the linearized problem at u . The solution u is said to be degenerate if one of the eigenvalues of the linearized problem is equal to zero. Otherwise it is called nondegenerate. [These definitions also hold for solutions of (1)].

Lemma 2.4 (Necessary and sufficient condition for the stability of solutions of (5)). *A solution of (5) is stable iff it is a nonnegative solution of (5) with maximum strictly greater than M .*

Proof. Suppose u is a nonnegative solution of (5) whose maximum is strictly greater than M . If we multiply both sides of (5) by v , both sides of (8) by u , subtract and integrate over Ω , we obtain

$$\int (f'(u)u - f(u))v = \mu \int uv.$$

The hypotheses imply $f'(u)u - f(u) \geq 0$. On the other hand $f'(u)u - f(u) \not\equiv 0$ because $\max_{\Omega} u > M$, $f(u) = 0$ for $u \leq M$, $f(u) > 0$ for $u > M$, f is C^2 and $u = 0$ on $\partial\Omega$. Since v is strictly positive on Ω and u is nonnegative, $\mu > 0$.

Conversely, suppose u is a stable solution of (5). We multiply both sides of (5) by u^- and integrate over Ω to obtain

$$\int |\nabla u^-|^2 = a \int (u^-)^2.$$

Using this equality,

$$\mu \int (u^-)^2 \leq \int |\nabla u^-|^2 - a \int (u^-)^2 + \int f'(u)(u^-)^2 = 0,$$

or $u^- = 0$, because $\mu > 0$. So u is nonnegative. If $\max_{\Omega} u \leq M$, then u satisfies $\Delta u + au = 0$. This implies $a = \lambda_1$. But then u is not stable, it is degenerate. Therefore $\max_{\Omega} u > M$. \square

Proof of Theorem 2.1. Consider the function $F : \mathbb{R} \times \mathcal{H} \rightarrow L^p(\Omega)$, defined by

$$F(a, u) = \Delta u + au - f(u).$$

We know $F(a_{\dagger}(t), u_{\dagger}(t)) = 0$, in particular for $t \in]M, M + \delta[$. Since for these values of t the functions $u_{\dagger}(t)$ are positive with maxima strictly greater than M , and hence nondegenerate, by the Implicit Function Theorem, in a

neighborhood of $(a_{\dagger}(t), u_{\dagger}(t))$, the solutions of $F(a, u) = 0$ in $\mathbb{R} \times \mathcal{H}$ may also be written in the form $(a, u_{\dagger}(a))$. Of course, in rigor $u_{\dagger}(t)$ should be called something else like $\tilde{u}_{\dagger}(t)$, as it is not $u_{\dagger}(a)$ but rather $\tilde{u}_{\dagger}(t) = u_{\dagger}(a_{\dagger}(t))$. However, there is no risk of confusion.

Using the argument in [13, proof of Theorem 2.5], one may extend \mathcal{C}_{\dagger} of Lemma 2.3 with a curve (still called \mathcal{C}_{\dagger}) of solutions $(a, u_{\dagger}(a))$ of (5) defined for $a \in]\lambda_1, +\infty[$. By the maximum principle, $u'_{\dagger}(a) > 0$. Let K_a satisfy

$$K_a > 0 \quad \text{and} \quad aK_a - f(K_a) = 0. \quad (9)$$

Note $au - f(u) \leq 0$ for $u > K_a$ because the hypotheses imply $u \mapsto f(u)/u$ is increasing. It is well known the maximum principle implies $u_{\dagger}(a) \leq K_a$ and then Hopf's Lemma [11, Lemma 3.4] implies $u_{\dagger}(a) < K_a$. This will be used in the proof of Proposition 4.2.

Let

$$t_{\dagger}(a) := \frac{\int u_{\dagger}(a)\phi}{\int \phi^2}. \quad (10)$$

Clearly, $\lim_{a \searrow \lambda_1} t_{\dagger}(a) = M$. We remark that t_{\dagger} is strictly increasing along \mathcal{C}_{\dagger} as

$$t'_{\dagger}(a) = \frac{\int u'_{\dagger}(a)\phi}{\int \phi^2} > 0.$$

The next lemma completes the proof of Theorem 2.1. \square

Lemma 2.5 (Uniqueness of stable solutions of (5)). *For each $a \in]\lambda_1, +\infty[$, $u_{\dagger}(a)$ is the unique stable solution of equation (5).*

Proof. As mentioned in the Introduction, the argument below gives a direct proof of the uniqueness part of Theorem 2.5 of [13], which does not use [1, Lemma 3.3]. The emphasis in [13] is on uniqueness of positive solutions and here is on uniqueness of stable solutions. Of course, these are the same via Lemma 2.4.

Let $\tilde{u}(a)$ satisfy $F(a, \tilde{u}(a)) = 0$ and be stable. We proved in Lemma 2.4 $\tilde{u}(a)$ is nonnegative with maximum strictly greater than M . Let us prove we may use the Implicit Function Theorem to follow the solution $\tilde{u}(a)$ as a decreases all the way down to λ_1 . The solutions $\tilde{u}(a)$ will not blow up to $+\infty$ as $\tilde{u}(a) < K_a$. And they will remain bounded below by zero as long as they remain stable. We now show the solutions $\tilde{u}(a)$ will remain stable for $a > \lambda_1$. Suppose $a_n \searrow a_0 > \lambda_1$ and $\tilde{u}(a_n)$ are stable solutions. Multiplying both sides of (5) by $\tilde{u}(a_n)$ and integrating over Ω ,

$$\int |\nabla \tilde{u}(a_n)|^2 = a_n \int (\tilde{u}(a_n))^2 - \int f(\tilde{u}(a_n))\tilde{u}(a_n). \quad (11)$$

The sequence $(\tilde{u}(a_n))$ is bounded in $L^\infty(\Omega)$ and in $H_0^1(\Omega)$. We may assume $\tilde{u}(a_n) \rightarrow u_0$ in $H^1(\Omega)$, $\tilde{u}(a_n) \rightarrow u_0$ in $L^2(\Omega)$ and $\tilde{u}(a_n) \rightarrow u_0$ a.e. in Ω . One

easily sees the function u_0 is a nonnegative solution of (5). If u_0 is not stable it must be less than or equal to M .

Let $w(a_n)$ be a first eigenfunction of the linearized equation at $\tilde{u}(a_n)$,

$$\Delta w(a_n) + a_n w(a_n) - f'(\tilde{u}(a_n))w(a_n) = -\mu(a_n)w(a_n),$$

normalized so $\int [w(a_n)]^2 = \int \phi^2$. The sequence $(w(a_n))$ is bounded in $H_0^1(\Omega)$. So we may also assume $w(a_n) \rightharpoonup w_0$ in $H_0^1(\Omega)$, $w(a_n) \rightarrow w_0$ in $L^2(\Omega)$ and $w(a_n) \rightarrow w_0$ a.e. in Ω . Moreover, by the Dominated Convergence Theorem $\int f'(\tilde{u}(a_n))w_0 \rightarrow \int f'(u_0)w_0$. Since, by the Implicit Function Theorem, $(\mu(a_n))$ must decrease to zero if u_0 is not stable,

$$\Delta w_0 + a_0 w_0 - f'(u_0)w_0 = 0,$$

with $w_0 \geq 0$. In fact with $w_0 > 0$. But $u_0 \leq M$ so $a_0 = \lambda_1$. This completes the proof that we can follow the branch $(a, \tilde{u}(a))$ all the way down to λ_1 .

Let $a_n \searrow \lambda_1$. The sequence $(\tilde{u}(a_n))$ is uniformly bounded. The fact $\tilde{u}(a_n)$ satisfy (5) and [11, Lemma 9.17] imply the norms $\|\tilde{u}(a_n)\|_{\mathcal{H}}$ are uniformly bounded. Thus $(\tilde{u}(a_n))$ has a strongly convergent subsequence in $L^p(\Omega)$. Subtracting equations (5) for $\tilde{u}(a_n)$ and $\tilde{u}(a_m)$ and using [11, Lemma 9.17], $(\tilde{u}(a_n))$ has a subsequence which is strongly convergent in \mathcal{H} .

It is easy to see using (7) the solutions of $F(\lambda_1, u) = 0$ are $u = t\phi$ where $t \in]-\infty, M]$, as for those solutions $\int f(u)\phi = 0$, and we already showed the only branches of solutions of (5) reaching $a = \lambda_1$ are \mathcal{C}_\dagger and the line \mathcal{A} . This implies $\tilde{u}(a) = u_\dagger(a)$ and completes the proof of the lemma. \square

3 Degenerate solutions with Morse index equal to zero

We now turn to (1). In Theorem 3.1 we may relax assumption **(a)** to

(a)' $h \in L^p(\Omega)$.

Here our main result is

Theorem 3.1 (\mathcal{D}_* , degenerate solutions with Morse index equal to zero). *Suppose f satisfies **(i)**-**(iv)** and h satisfies **(a)'**, **(b)**. The set of degenerate solutions (a, u, c) of (1) with Morse index equal to zero is a connected one dimensional manifold \mathcal{D}_* of class C^1 in $\mathbb{R} \times \mathcal{H} \times \mathbb{R}$. The manifold is the union of the half line $\{(\lambda_1, t\phi, 0) : t \in]-\infty, M]\}$ with a graph $\{(a, u_*(a), c_*(a)) : a \in]\lambda_1, +\infty[\}$. The function c_* is nonnegative.*

The half line $(\lambda_1, t\phi, 0)$ for $t \in]-\infty, M]$ consists of degenerate solutions of (1) with Morse index equal to zero. Consider the function $G : \mathbb{R}^2 \times \mathcal{R} \times \mathbb{R} \times S \rightarrow L^p(\Omega) \times L^p(\Omega)$, where

$$S = \{w \in \mathcal{H} : \int w^2 = \int \phi^2\}, \quad (12)$$

G defined by

$$\begin{aligned} G(a, t, y, c, w) = & (\Delta(t\phi + y) + a(t\phi + y) - f(t\phi + y) - ch, \\ & \Delta w + aw - f'(t\phi + y)w). \end{aligned}$$

Then $G(\lambda_1, t, 0, 0, \phi) = 0$ for $t \in]-\infty, M]$. Notice the difference in notation, S in (4) as opposed to S in (12).

Lemma 3.2 (Initial portion of \mathcal{D}_*). *There exists $\sigma > 0$ and C^1 functions $a_* : J \rightarrow \mathbb{R}$, $y_* : J \rightarrow \mathcal{R}$, $c_* : J \rightarrow \mathbb{R}$ and $w_* : J \rightarrow S$, where $J =]-\infty, M + \sigma[$, such that the map $t \mapsto (a_*(t), t\phi + y_*(t), c_*(t))$, defined in J , with $a_*(t) = \lambda_1$, $y_*(t) = 0$, $c_*(t) = 0$, $w_*(t) = \phi$ for $t \in]-\infty, M]$, parametrizes a curve \mathcal{D}_* of degenerate solutions of (1) with Morse index equal to zero. There exists a neighborhood of \mathcal{D}_* in $\mathbb{R} \times \mathcal{H} \times \mathbb{R}$ such that the degenerate solutions of the equation in this neighborhood lie on \mathcal{D}_* . The degenerate directions are given by w_* .*

Proof. Let $t_0 \in]-\infty, M]$. We use the Implicit Function Theorem to show we may write the solutions of $G(a, t, y, c, w) = 0$, in a neighborhood of $(\lambda_1, t_0, 0, 0, \phi)$, in the form $(a_*(t), t, y_*(t), c_*(t), w_*(t))$. Let $(\alpha, z, \gamma, \omega) \in \mathbb{R} \times \mathcal{R} \times \mathbb{R} \times \mathcal{R}$. The derivative $DG_{(a, y, c, w)}(\alpha, z, \gamma, \omega)$ at $(\lambda_1, t_0, 0, 0, \phi)$ is

$$\begin{aligned} G_a \alpha + G_y z + G_c \gamma + G_w \omega &= (\alpha(t\phi + y) + \Delta z + az - f'(t\phi + y)z - \gamma h, \\ &\quad \alpha\omega - f''(t\phi + y)zw + \Delta\omega + a\omega - f'(t\phi + y)\omega) \\ &= (\alpha t_0 \phi + \Delta z + \lambda_1 z - \gamma h, \alpha\phi + \Delta\omega + \lambda_1 \omega). \end{aligned}$$

We check this derivative is injective. Consider the system obtained by setting the previous derivative equal to $(0, 0)$. Multiplying both sides of the second equation by ϕ and integrating we get $\alpha = 0$. Thus $\Delta\omega + \lambda_1 \omega = 0$. Since $\omega \in \mathcal{R}$ it follows $\omega = 0$. Now we use the first equation. Multiplying both sides of $\Delta z + \lambda_1 z - \gamma h = 0$ by ϕ and integrating we get $\gamma = 0$. This implies $z = 0$ and proves injectivity. It is easy to check that the derivative is also surjective. So the derivative is a homeomorphism from $\mathbb{R} \times \mathcal{R} \times \mathbb{R} \times \mathcal{R}$ to $L^p(\Omega) \times L^p(\Omega)$. \square

Let σ be as in Lemma 3.2. Because y_* is C^1 and $y'_*(M) = 0$, reducing σ if necessary,

$$u_*(t) := t\phi + y_*(t)$$

is positive and $\max_{\Omega} u_*(t) > M$ for $t \in]M, M + \sigma[$. Moreover, for t in this interval, $a_*(t) > \lambda_1$ and $c_*(t) > 0$ because

$$(a - \lambda_1) \int w \phi = \int f'(u) w \phi$$

and

$$\int (f'(u)u - f(u))w = c \int hw. \quad (13)$$

We now continue the branch of degenerate solutions \mathcal{D}_* using a as a parameter.

Proof of Theorem 3.1. Consider the function $H : \mathbb{R} \times \mathcal{H} \times \mathbb{R} \times S \rightarrow L^p(\Omega) \times L^p(\Omega)$, defined by

$$H(a, u, c, w) = (\Delta u + au - f(u) - ch, \Delta w + aw - f'(u)w).$$

We know $H(a_*(t), u_*(t), c_*(t), w_*(t)) = 0$, in particular for $t \in]M, M + \sigma[$. We use the Implicit Function Theorem to show that we may write the solutions of $H(a, u, c, w) = 0$, in a neighborhood of a solution (a_0, u_0, c_0, w_0) , with $a_0 > \lambda_1$, in the form $(a, u_*(a), c_*(a), w_*(a))$. [As above, $u_*(t)$ should be called $\tilde{u}_*(t)$ where $\tilde{u}_*(t) = u_*(a_*(t))$, and similarly for $c_*(t)$ and $w_*(t)$]. Let $(v, \gamma, \omega) \in \mathcal{H} \times \mathbb{R} \times \mathcal{R}_{w_0}$, with

$$\mathcal{R}_w := \{\omega \in \mathcal{H} : \int \omega w = 0\}.$$

The derivative $DH_{(u,c,w)}(v, \gamma, \omega)$ at (a_0, u_0, c_0, w_0) is

$$\begin{aligned} H_u v + H_c \gamma + H_w \omega &= (\Delta v + a_0 v - f'(u_0)v - \gamma h, \\ &\quad -f''(u_0)vw_0 + \Delta \omega + a_0 \omega - f'(u_0)\omega). \end{aligned}$$

We check that the derivative is injective. Consider the system obtained by setting the previous derivative equal to $(0, 0)$. Multiplying both sides of the first equation by w_0 and integrating by parts we obtain $\gamma = 0$. Thus $\Delta v + a_0 v - f'(u_0)v = 0$. So $v = \kappa w_0$ where $\kappa \in \mathbb{R}$. Substituting into the second equation,

$$-\kappa f''(u_0)w_0^2 + \Delta \omega + a_0 \omega - f'(u_0)\omega = 0.$$

Multiplying both sides of this equation by w_0 and integrating by parts it follows

$$\kappa \int f''(u_0)w_0^3 = 0.$$

This implies that either $\kappa = 0$ or $f''(u_0) \equiv 0$. The function $f''(u_0)$ is identically zero iff $u_0 \leq M$. In this case $\Delta w_0 + a_0 w_0 = 0$. Then $a_0 = \lambda_1$ because $w_0 > 0$. So $\kappa = 0$ and $\Delta \omega + a_0 \omega - f'(u_0) \omega = 0$. The function ω has to be a multiple of w_0 . On the other hand, $\omega \in \mathcal{R}_{w_0}$. Finally, $\omega = 0$. We have proved injectivity. It is easy to check that the derivative is also surjective. So the derivative is a homeomorphism from $\mathcal{H} \times \mathbb{R} \times \mathcal{R}_{w_0}$ to $L^p(\Omega) \times L^p(\Omega)$.

The branch of degenerate solutions $(a_*(t), t\phi + y_*(t), c_*(t))$ for $t \in]M, M + \sigma[$ may also be represented as $(a, u_*(a), c_*(a))$. The inverse of $a_*|_{]M, M + \sigma[}$ is t_* , with

$$t_*(a) := \frac{\int u_*(a) \phi}{\int \phi^2}. \quad (14)$$

It follows that $t_*(a)$ is an increasing function of a for $a \in]\lambda_1, \lim_{t \nearrow M + \sigma} a_*(t)[$.

Equality (13) shows that for any degenerate solution with Morse index equal to zero the value of c is nonnegative. As $c_*(a) \geq 0$, we have $\max_{\Omega} u_*(a) \leq K_a$ (K_a as in (9)). Moreover, we just saw that $\max_{\Omega} u_*(a)$ cannot drop to M for $a > \lambda_1$. We may follow the branch of degenerate solutions $(a, u_*(a), c_*(a), w_*(a))$ while $c_*(a)$ does not go to $+\infty$ and $\min_{\Omega} u_*(a)$ does not go to $-\infty$. Suppose the branch of degenerate solutions exists for $a \in]\lambda_1, L[$. Let $a_n \nearrow L$. We wish to prove that, modulo a subsequence, $\lim_{n \rightarrow +\infty} u_*(a_n)$ and $\lim_{n \rightarrow +\infty} c_*(a_n)$ exist. This will imply that the branch can be extended for all $a \in]\lambda_1, +\infty[$.

By a computation similar to the one leading to (11), $[u_*(a_n)]^+$ is bounded in $H_0^1(\Omega)$. Modulo a subsequence, $[u_*(a_n)]^+ \rightharpoonup u_L$ in $H_0^1(\Omega)$, $[u_*(a_n)]^+ \rightarrow u_L$ in $L^2(\Omega)$ and $[u_*(a_n)]^+ \rightarrow u_L$ a.e. in Ω . Arguing as in the proof of Lemma 2.5, $w_*(a_n) \rightharpoonup w_L$ in $H_0^1(\Omega)$ where w_L satisfies $\Delta w_L + L w_L - f'(u_L) w_L = 0$, and so $w_L > 0$. Differentiating both sides of $H(a, u_*(a), c_*(a), w_*(a)) = 0$ with respect to a , we readily deduce

$$c'_*(a) = \frac{\int u_*(a) w_*(a)}{\int h w_*(a)}.$$

Since the sequence $(u_*(a_n))$ is uniformly bounded above and w_L is positive, $\limsup_{n \rightarrow \infty} c'_*(a_n)$ is finite. Recalling that c_* is nonnegative, we conclude that $\lim_{n \rightarrow \infty} c_*(a_n)$ exists. We call the limit c_L .

Next we show $\|u_*(a_n)\|_{L^2(\Omega)}$ are uniformly bounded. By contradiction suppose $\|u_*(a_n)\|_{L^2(\Omega)} \rightarrow +\infty$. Define $v_*(a_n) = u_*(a_n) / \|u_*(a_n)\|_{L^2(\Omega)}$. The new functions satisfy

$$\Delta v_*(a_n) + a_n v_*(a_n) - \frac{f(u_*(a_n))}{\|u_*(a_n)\|_{L^2(\Omega)}} - \frac{c_*(a_n)}{\|u_*(a_n)\|_{L^2(\Omega)}} h = 0.$$

Multiplying both sides by $v_*(a_n)$ and integrating we conclude $(v_*(a_n))$ is bounded in $H_0^1(\Omega)$. Modulo a subsequence, $v_*(a_n) \rightharpoonup v$ in $H_0^1(\Omega)$, $v_*(a_n) \rightarrow v$

in $L^2(\Omega)$ and $v_*(a_n) \rightarrow v$ a.e. in Ω , where

$$\Delta v + Lv = 0.$$

The function v is nonpositive because $(u_*(a_n))$ is uniformly bounded above. The function v is negative because it has $L^2(\Omega)$ norm equal to one. So $L = \lambda_1$, which is a contradiction. For use below, we observe that even in the case $L = \lambda_1$ we are lead to a contradiction, as we now see. Indeed, [11, Lemma 9.17] implies $\|v_*(a_n)\|_{\mathcal{H}}$ are uniformly bounded. We may assume $v_*(a_n) \rightarrow v$ in $C^{1,\alpha}(\bar{\Omega})$. Hopf's lemma implies $v_*(a_n)$, and hence $u_*(a_n)$, are negative for large n . For these n , the linearized equations become

$$\Delta w_*(a_n) + a_n w_*(a_n) = 0.$$

This is a contradiction as $a_n > \lambda_1$ and the $w_*(a_n)$ are positive. We have proved $\|u_*(a_n)\|_{L^2(\Omega)}$ are uniformly bounded.

We use the fact that $(a, u_*(a), c_*(a))$ satisfies (1) and, once more, [11, Lemma 9.17] to guarantee that, modulo a subsequence, $(u_*(a_n))$ converges strongly in \mathcal{H} to a function we designate by $u_*(L)$, as it satisfies (1) with $a = L$ and $c = c_L$. This finishes the proof that the branch of degenerate solutions can be extended to all $a > \lambda_1$, because we may apply the Implicit Function Theorem to $H = 0$ at the solution $(L, u_*(L), c_L, w_L)$.

To finish the proof of Theorem 3.1, we notice that if $(a, \tilde{u}(a), \tilde{c}(a))$ is a degenerate solution of (1) with Morse index equal to zero, then, again by the Implicit Function Theorem, we can follow this solution backwards using the parameter a until we reach λ_1 . Let $a_n \searrow \lambda_1$. The norms $\|\tilde{u}(a_n)\|_{L^2(\Omega)}$ are uniformly bounded, as we saw two paragraphs above. Arguing as before, $\tilde{c}(a_n) \rightarrow \tilde{c}_0$. The sequence $(a_n, \tilde{u}(a_n), \tilde{c}(a_n))$ will converge in $\mathbb{R} \times \mathcal{H} \times \mathbb{R}$ to, say, $(\lambda_1, \tilde{u}_0, \tilde{c}_0)$, solution of (1) (see the end of the proof of Lemma 2.5). Multiplying both sides of (1) by ϕ and integrating $\tilde{c}_0 = 0$. By Lemma 3.2 we have $\tilde{u}_0 = M\phi$ and so $\tilde{u}(a) = u_*(a)$. \square

Remark 3.3. Assume **(a)**. If (a_n, u_n, c_n) is a sequence of solutions of (1), with $a_n > \lambda_1$ and bounded away from λ_1 , (a_n) bounded above, and (c_n) bounded, then (u_n) is uniformly bounded. If $c_n \geq 0$ the same is true under the weaker assumption **(a)'**.

Indeed, the hypotheses in the remark imply (u_n) is uniformly bounded above. Therefore we may argue as in the proof of Theorem 3.1.

Proposition 3.4 (Behavior of c along \mathcal{D}_*). Suppose f satisfies **(i)**-**(iv)** and $h \in \{h \in C^1(\bar{\Omega}) : h = 0 \text{ on } \partial\Omega\}$ satisfies **(b)**. Along \mathcal{D}_* we have

$$\lim_{a \rightarrow +\infty} c_*(a) = +\infty.$$

Proof. Suppose $\underline{c} > 0$. Fix any $l > \lambda_1$. Choose $t > 0$ such that $t\phi \leq u_{\dagger}(l)$ where $(l, u_{\dagger}(l))$ is the stable solution of equation (5) in Section 2. Such a t exists by (6). Next choose $A \geq l$ large enough satisfying

$$(A - \lambda_1)t\phi - f(t\phi) - \underline{c}h \geq 0.$$

Take $a > A$. Then $(a, t\phi, \underline{c})$ is a subsolution of (1). And $(a, u_{\dagger}(a), \underline{c})$ is a supersolution of (1). The subsolution and the supersolution are ordered, $t\phi \leq u_{\dagger}(l) < u_{\dagger}(a)$, as the solutions u_{\dagger} are strictly increasing along \mathcal{C}_{\dagger} . Moreover, neither $(a, t\phi, \underline{c})$ nor $(a, u_{\dagger}(a), \underline{c})$ is a solution of (1). Let K_a be as in (9). Clearly, $u_{\dagger}(a) \leq K_a$. Define

$$f_{K_a}(u) = \begin{cases} f(u) & \text{if } u \leq K_a, \\ f(K_a) + f'(K_a)(u - K_a) & \text{if } u > K_a. \end{cases} \quad (15)$$

By [4, Theorem 2], there is a solution $u_0 \in H_0^1(\Omega)$ of

$$\Delta u + au - f_{K_a}(u) - \underline{c}h = 0, \quad (16)$$

$t\phi < u_0 < u_{\dagger}(a)$, such that u_0 is a local minimum of $I_{K_a} : H_0^1(\Omega) \rightarrow \mathbb{R}$, defined by

$$I_{K_a}(u) := \frac{1}{2} \int (|\nabla u|^2 - au^2) + \int F_{K_a}(u) + \underline{c} \int hu. \quad (17)$$

Here $F_{K_a}(u) = \int_0^u f_{K_a}(s) ds$. By [11, Lemma 9.17] $u_0 \in \mathcal{H}$. Since $\mathcal{H} \subset H_0^1(\Omega)$, differentiating I_{K_a} twice, u_0 is either stable or degenerate with Morse index equal to zero in \mathcal{H} . As $u_0 < u_{\dagger}(a) < K_a$, and f_{K_a} (respectively f'_{K_a}) coincides with f (respectively f') below K_a , u_0 is a solution of (1), either stable or degenerate with Morse index equal to zero. By the Theorem 4.1 ahead, (a, u_0, \underline{c}) must lie on $\mathcal{M}^* \cup \{\mathbf{p}_*\}$ and $c_*(a) \geq \underline{c}$. We have shown that given $\underline{c} > 0$, there exists $A > 0$, such that for all $a > A$ we have $c_*(a) \geq \underline{c}$. The proof is complete. \square

4 Stable solutions, solutions around \mathcal{D}_* , solutions around zero, and mountain pass solutions

In this section we treat successively stable solutions in Theorem 4.1 and Proposition 4.2, solutions around \mathcal{D}_* in Lemma 4.3, solutions around zero in Lemma 4.4, and mountain pass solutions in Lemma 4.5. We finish by proving the existence of at least three solutions for $a > \lambda_2$, a not an eigenvalue, and small $|c|$ in Theorem 4.6.

Theorem 4.1 (\mathcal{M}^* , stable solutions of (1)). *Suppose f satisfies (i)-(iv) and h satisfies (a)-(b). Fix $a > \lambda_1$. The set of stable solutions (c, u) of (1) is a C^1 manifold \mathcal{M}^* , which is the graph $\{(c, u^*(c)) : c \in]-\infty, c_*[\}$. The limit $\lim_{c \nearrow c_*} (c, u^*(c))$ exists and equals $\mathbf{p}_* := (c_*, u_*)$, the degenerate solution with Morse index equal to zero on the curve \mathcal{D}_* (i.e. $(c_*, u_*) = (c_*(a), u_*(a))$) of Theorem 3.1). If $c_1 < c_2$, then $u^*(c_1) > u^*(c_2)$.*

Sketch of the proof. [§] The argument is very close to [13, proof of Theorem 3.2]. Consider the function $\tilde{H} : \mathcal{H} \times \mathbb{R} \times S \times \mathbb{R} \rightarrow L^p(\Omega) \times L^p(\Omega)$ (S given in (12)), \tilde{H} defined by

$$\tilde{H}(u, c, w, \mu) = (\Delta u + au - f(u) - ch, \Delta w + aw - f'(u)w + \mu w).$$

Apply the Implicit Function Theorem to describe the solutions of $\tilde{H} = 0$ in a neighborhood of a stable solution (u, c, w, μ) . Here μ is the first eigenvalue of the associated linearized problem and w is the corresponding positive eigenfunction on S . For each fixed a , the solutions of $\tilde{H} = 0$ in a neighborhood of a stable solution (u, c, w, μ) lie on a C^1 curve parametrized by $c \mapsto (u^*(c), c, w^*(c), \mu^*(c))$. Differentiate both sides of the equations $\tilde{H}(u^*(c), c, w^*(c), \mu^*(c)) = (0, 0)$ with respect to c . When $\mu > 0$, using the maximum principle,

$$(u^*)' < 0$$

and

$$(\mu^*)' = \frac{\int f''(u^*)(u^*)'(w^*)^2}{\int \phi^2} < 0.$$

By Remark 3.3, we may follow the solution $u^*(c)$ until it becomes degenerate. The solution $u^*(c)$ will have to become degenerate for some value of c . Indeed, from (1) we obtain

$$c \int h\phi = (a - \lambda_1) \int u\phi - \int f(u)\phi \leq (a - \lambda_1) \int u^+\phi, \quad (18)$$

showing c is bounded above. Thus, the solutions $u^*(c)$ cannot be continued for all positive values of c . There must exist c_* such that $\lim_{c \nearrow c_*} \mu^*(c) = 0$. Clearly, the solutions $u^*(c)$ will converge to a solution u_* as $c \nearrow c_*$. By the uniqueness assertion in Theorem 3.1, this (c_*, u_*) belongs to \mathcal{D}_* . In particular $c_* > 0$.

The above shows any branch of stable solutions can be extended for $c \in]-\infty, c_*[$. But by Lemma 2.5 there is a unique stable solution of (1) for $c = 0$, namely $u_{\dagger} = u_{\dagger}(a)$. This proves uniqueness. \square

[§]For the full proof see the Appendix 9.

Proposition 4.2 (Stable solutions are superharmonic for small $|c|$). *Suppose f satisfies (i)-(iv) and $h \in \{\bar{h} \in C^1(\bar{\Omega}) : \bar{h} = 0 \text{ on } \partial\Omega\}$ satisfies (b). Let $a > \lambda_1$ and u^* be as in Theorem 4.1. For small $|c|$,*

$$au^*(c) - f(u^*(c)) > ch. \quad (19)$$

This generalizes formula (3.2) of [13]. The proof of [13] does not carry through in our setting however.

Proof of Proposition 4.2. Fix $a > \lambda_1$ and let $u_\dagger = u_\dagger(a)$ be the stable solution of (1) for $c = 0$. In the proof of Theorem 2.1 we showed $0 < u_\dagger < K_a$. So $au_\dagger - f(u_\dagger)$ is strictly positive on Ω . Hopf's lemma implies $\frac{\partial u_\dagger}{\partial n} < 0$ on $\partial\Omega$, where n denotes the exterior unit normal to $\partial\Omega$. As $f'(0) = 0$, we also have $\frac{\partial}{\partial n}(au_\dagger - f(u_\dagger)) < 0$ on $\partial\Omega$. Since the function u^* in Theorem 4.1 is C^1 into \mathcal{H} , and so in particular continuous, and \mathcal{H} is compactly embedded in $C^{1,\alpha}(\bar{\Omega})$, it follows $\frac{\partial}{\partial n}(au^*(c) - f(u^*(c))) < 0$ on $\partial\Omega$ for small $|c|$. Moreover, $au^*(c) - f(u^*(c)) > 0$ in Ω for small $|c|$. If h belongs to the space $C^1(\bar{\Omega})$ and vanishes on $\partial\Omega$, then we obtain (19), for small $|c|$. \square

Lemma 4.3 (Solutions around \mathcal{D}_* , [10, Theorem 3.2], [13, p. 3613]). *Let $a > \lambda_1$ be fixed and $\mathbf{p}_* = (c_*, u_*)$ be a degenerate solution with Morse index equal to zero. There exists a neighborhood of \mathbf{p}_* in $\mathbb{R} \times \mathcal{H}$ such that the set of solutions of (1) in the neighborhood is a C^1 manifold. This manifold is $\mathbf{m}^\sharp \cup \{\mathbf{p}_*\} \cup \mathbf{m}^*$. Here*

- \mathbf{m}^\sharp is a manifold of nondegenerate solutions with Morse index equal to one, which is a graph $\{(c, u^\sharp(c)) : c \in]c_* - \varepsilon_*, c_*[\}$.
- \mathbf{m}^* is a manifold of stable solutions, which is a graph $\{(c, u^*(c)) : c \in]c_* - \varepsilon_*, c_*[\}$.

The value ε_* is positive. The manifolds \mathbf{m}^\sharp and \mathbf{m}^* are connected by $\{\mathbf{p}_*\}$.

Sketch of the proof. ¶ This lemma is known, but we sketch a proof adapted to our framework. Let (c_*, u_*) be a degenerate solution with Morse index equal to zero. Let t_* and y_* be such that $u_* = t_* w_* + y_*$, with $w_* \in S$ satisfying $\Delta w_* + a w_* - f'(u_*) w_* = 0$, $w_* > 0$, and $y_* \in \mathcal{R}_{w_*}$. We let $\tilde{G} : \mathbb{R} \times \mathcal{R}_{w_*} \times \mathbb{R} \times S \times \mathbb{R} \rightarrow L^p(\Omega) \times L^p(\Omega)$ be defined by

$$\begin{aligned} \tilde{G}(t, y, c, w, \mu) &= (\Delta(tw_* + y) + a(tw_* + y) - f(tw_* + y) - ch, \\ &\quad \Delta w + aw - f'(tw_* + y)w + \mu w). \end{aligned}$$

¶For the full proof see the Appendix 9.

We may use the Implicit Function Theorem to describe the solutions of $\tilde{G} = 0$ in a neighborhood of $(t_*, y_*, c_*, w_*, 0)$. They lie on a curve $t \mapsto (t, y(t), c(t), w(t), \mu(t))$. It is impossible for $\max_{\Omega}(t_* w_* + y_*) \leq M$ because otherwise $\Delta w_* + a w_* = 0$ with $w_* > 0$. Differentiating

$$\tilde{G}(t, y(t), c(t), w(t), \mu(t)) = (0, 0)$$

with respect to t ,

$$\mu'(t_*) = \frac{\int f''(t_* w_* + y_*) w_*^3}{\int w_*^2} > 0,$$

$c'(t_*) = 0$ and

$$c''(t_*) = - \frac{\int f''(t_* w_* + y_*) w_*^3}{\int h w_*} < 0.$$

This is formula (2.7) of [13]. We recall from the proof of Theorem 3.1, a degenerate solution with $a > \lambda_1$ has $c_* > 0$. As t increases from t_* , $c(t)$ decreases and the solution becomes stable. So the “end” of \mathcal{M}^* coincides with the piece of curve parametrized by $t \mapsto (c(t), t w_* + y(t))$, for t in a right neighborhood of t_* . A parametrization of \mathbf{m}^\sharp is obtained by taking t in a left neighborhood of t_* . \square

Let $-\infty =: \lambda_0 < \lambda_1 < \lambda_2 < \dots < \lambda_i < \dots$, denote the eigenvalues of the Dirichlet Laplacian on Ω .

Lemma 4.4 (Solutions around zero, [13, Theorem 3.3], [6, Theorem 2]). *Let $\lambda_i < a < \lambda_{i+1}$ for some $i \geq 0$. There exists a C^1 function \check{u} defined in $] -\check{c}, \check{c}[$ such that $c \mapsto (c, \check{u}(c))$ parametrizes a curve \check{C} of solutions of (1) passing at $(0, 0)$. The solutions on \check{C} have Morse index equal to the sum of the dimensions of the eigenspaces corresponding to λ_1 to λ_i . If a is sufficiently close to λ_1 , $c_1 < c_2$ implies $\check{u}(c_1) < \check{u}(c_2)$.*

We refer to the cited works for the proof of this lemma.^{||} The assertion on the Morse index follows from its continuity on \check{u} .

Lemma 4.5 (Mountain pass solutions, [5]). *Let $a > \lambda_1$ and $c < c_*(a)$. Then (1) has at least two solutions, the stable solution and a mountain pass solution.*

Proof. Choose $K = K(a, c)$ such that

$$f'(K) > a \quad \text{and} \quad (u \geq K \Rightarrow au - f(u) - ch \leq 0 \text{ in } \Omega). \quad (20)$$

^{||}For the full proof see Appendix 9.

Define f_K by (15) with K_a replaced by K . By the maximum principle, (a, u, c) is a solution of (1) iff (a, u, c) is a solution of (16) (with f_{K_a} replaced by f_K and \underline{c} replaced by c). It is easy to see that I_K , defined by (17), satisfies the Palais-Smale condition. Indeed, $I'_K(u_n) \rightarrow 0$ implies $(\|u_n\|_{H_0^1(\Omega)})$ is bounded, and then $(I_K(u_n))$ convergent implies (u_n) has a convergent subsequence in $H_0^1(\Omega)$. Also, $I_K(t\phi) \rightarrow -\infty$ as $t \rightarrow -\infty$. As for the solution $(a, u^*(c), c)$ of (1) we have $u^*(c) \leq K$, and f (respectively f') coincides with f_K (respectively f'_K) below K , $(a, u^*(c), c)$ is a stable in \mathcal{H} solution of (16). It is also stable in $H_0^1(\Omega)$ because of [11, Lemma 9.17]. By the Mountain-Pass Lemma ([2, Theorem 2.1]), there exists a solution (a, u_1, c) of (16). By the maximum principle $u_1 \leq K$. Again because f_K coincides with f below K , (a, u_1, c) is a solution of (1). \square

Theorem 4.6 (Existence of at least three solutions for $\lambda_2 < a \neq \lambda_i$ and small $|c|$). *Suppose f satisfies (i)-(iv) and h satisfies (a). If $\lambda_i < a < \lambda_{i+1}$, for some $i \geq 2$, and $|c|$ is small, then (1) has at least three solutions.*

Note here we do not need to assume h satisfies (b).

Proof of Theorem 4.6. Let $c < c_*(a)$. From [14, Theorem 10.15], the Morse index of (a, u_1, c) is at most equal to one in $H_0^1(\Omega)$, and (a, u_1, c) is degenerate if it has Morse index equal to zero. In the second case the Morse index of (a, u_1, c) in \mathcal{H} is zero and this solution is either stable or degenerate in \mathcal{H} . But it cannot be stable in \mathcal{H} because our analysis implies that, for fixed a , there is at most one stable solution for each c and $u_1 \neq u^*(c)$. It cannot be degenerate either because $c < c_*(a)$. We conclude the Morse index of (a, u_1, c) in \mathcal{H} has to be equal to one.

Let $|c| < \min\{\check{c}, c_*\}$. Then (1) has the stable solution (with Morse index equal to zero), the mountain pass solution (with Morse index equal to one) and the solution \check{u} of Lemma 4.4 (with Morse index at least equal to 2). \square

In case $\lambda_1 < a < \lambda_2$, it follows from Theorem 1.1, which we will prove ahead, we have $u_1 = u^\sharp$. Furthermore, for small $|c|$, we have $u_1 = \check{u}$.

5 Global bifurcation below λ_2

In this section we obtain global bifurcation curves below λ_2 . We briefly treat the case $a \leq \lambda_1$. Then we examine the situation in a right neighborhood of λ_1 . Finally we prove Theorem 1.1.

Let (a, u, c) be a solution of (1). Consider the quadratic form

$$Q_a(v) := \int (|\nabla v|^2 - av^2 + f'(u)v^2). \quad (21)$$

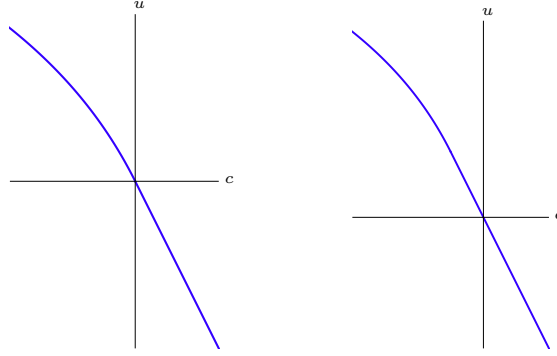


Figure 4: Bifurcation curve for $\lambda_1 - \delta < a < \lambda_1$. On the left $M = 0$ and on the right $M > 0$.

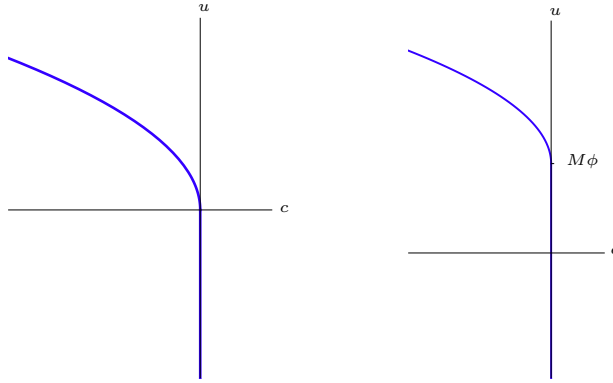


Figure 5: Bifurcation curve for $a = \lambda_1$. On the left $M = 0$ and on the right $M > 0$.

For any $v \in \mathcal{H}$ with $L^2(\Omega)$ with norm equal to one, $Q_a(v) \geq \lambda_1 - a$. If $a < \lambda_1$, then u is stable. The study of the bifurcation curves for $a \leq \lambda_1$ is simple. We only draw the final pictures in Figures 4 and 5. If one decreases a below $\lambda_1 - \delta$ the linear part of the curve for $c > 0$ starts bending. This evolution is similar to the one that happens from Figure 6 to Figure 1. We should mention the linearity of the branches for $c > 0$ in Figure 4 is a consequence of

$$u = c \left[\frac{\int h\phi}{a - \lambda_1} \phi + (\Delta + a)^{-1} \left[h - \left(\int h\phi \right) \phi \right] \right],$$

valid when the right hand side is less than or equal to M .

We remark that multiplying both sides of (1) by ϕ and integrating, we see that when u and c are both nonnegative, u not identically zero, then the constant a is greater than or equal to λ_1 .

We turn to the case $a > \lambda_1$.

Theorem 5.1 ($\lambda_1 \leq a < \lambda_1 + \delta$). Suppose f satisfies (i)-(iv) and h satisfies (a)'-(b). There exists $\delta > 0$ such that the following holds. The solutions (a, u, c) of (1) for $\lambda_1 \leq a < \lambda_1 + \delta$ and $c \geq 0$ can be parametrized in the global chart

$$\mathcal{I}_\phi = \{(a, t) \in \mathbb{R}^2 : \lambda_1 \leq a < \lambda_1 + \delta \text{ and } 0 \leq t \leq t_+(a)\}$$

(with $t_+(a)$ given in (10)) by

$$(a, t) \mapsto (a, u_\phi(a, t), c_\phi(a, t)), \quad u_\phi(a, t) = t\phi + y_\phi(a, t).$$

Here $y_\phi : \mathcal{I}_\phi \rightarrow \mathcal{R}$ and $c_\phi : \mathcal{I}_\phi \rightarrow \mathbb{R}$ are C^1 functions. We have $u_\phi(\lambda_1, t) = t\phi$ and $c_\phi(\lambda_1, t) = 0$, for $t \in [0, t_+(\lambda_1)] = [0, M]$; in addition $u_\phi(a, 0) = 0$, $c_\phi(a, 0) = 0$, $u_\phi(a, t_+(a))$ is the stable solution $u_+(a)$ of (5), and $c_\phi(a, t_+(a)) = 0$, for $a \in]\lambda_1, \lambda_1 + \delta[$. For each fixed a , the map $t \mapsto c_\phi(a, t)$ is strictly increasing until the corresponding solution lies on the degenerate curve \mathcal{D}_* of Section 3, and then is strictly decreasing until zero. The solutions $u_\phi(a, t)$ are strictly increasing with t , and so in particular are positive for $t \in]0, t_+(a)[$.

Remark 5.2. In Theorem 5.1 we may relax assumption (b) to

$$(b)'' \quad \int h\phi > 0.$$

Incidentally, this is also true for Lemma 4.4 (where this assumption is used only in connection with the last assertion).

Remark 5.3. When $M = 0$, as $a \searrow \lambda_1$ the curve $\{(c_\phi(a, t), u_\phi(a, t)) : t \in [0, t_+(a)]\}$ degenerates onto the point $(0, 0)$. This case was studied in [13] for f a quadratic function, as mentioned in the Introduction. When $M > 0$, as $a \searrow \lambda_1$ the curve $\{(c_\phi(a, t), u_\phi(a, t)) : t \in [0, t_+(a)]\}$ degenerates onto the segment $\{(0, t\phi) : t \in [0, M]\}$.

Remark 5.2 is in line with Theorem 1.3 of [3] ((b)'' corresponds to [3, formula (1.9)]). Theorem 5.1 is illustrated in Figure 6.

Proof of Theorem 5.1. We write $u = t\phi + y$, with $y \in \mathcal{R}$, and we construct a surface of solutions of (1) parametrized by a and t . Let

$$\tilde{g}(a, t, y, c) = \Delta(t\phi + y) + a(t\phi + y) - f(t\phi + y) - ch.$$

Let $t_0 \leq M$. Clearly $\tilde{g}(\lambda_1, t_0, 0, 0) = 0$. At $(\lambda_1, t_0, 0, 0)$,

$$\tilde{g}_y z + \tilde{g}_c \gamma = \Delta z + \lambda_1 z - \gamma h = 0$$

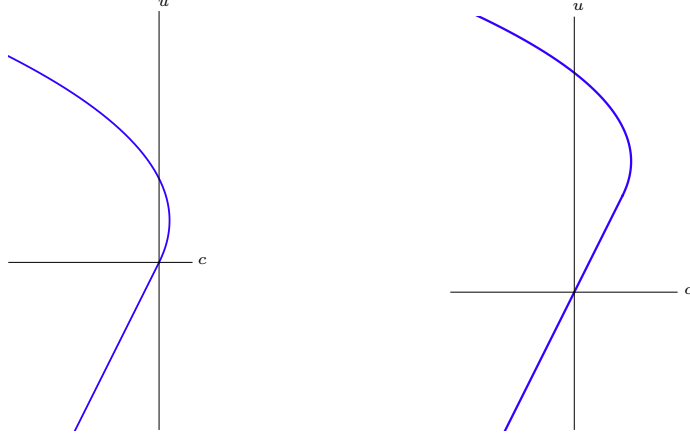


Figure 6: Bifurcation curve for $\lambda_1 < a < \lambda_1 + \delta$. On the left $M = 0$ and on the right $M > 0$.

implies $\gamma = 0$ and $z = 0$. The Implicit Function Theorem guarantees in a neighborhood of $(\lambda_1, t_0, 0, 0)$ the solutions of $\tilde{g} = 0$ lie on a surface $(a, t) \mapsto (a, t, y_\phi(a, t), c_\phi(a, t))$. Let now $\lambda_1 < a \neq \lambda_i$ for all $i > 1$. At $(a, 0, 0, 0)$,

$$\tilde{g}_y z + \tilde{g}_c \gamma = \Delta z + a z - \gamma h = 0$$

also implies $\gamma = 0$ and $z = 0$. The Implicit Function Theorem again guarantees in a neighborhood of $(a, 0, 0, 0)$ the solutions of $\tilde{g} = 0$ lie on a surface $(a, t) \mapsto (a, t, y_\phi(a, t), c_\phi(a, t))$. In particular, we have a surface of solutions defined in a neighborhood \mathcal{N} of

$$\{(a, t) \in \mathbb{R}^2 : (a = \lambda_1 \text{ and } 0 \leq t \leq M) \text{ or } (\lambda_1 \leq a < \lambda_2 \text{ and } t = 0)\},$$

such that the solutions on this surface are the unique solutions of the equation $\tilde{g} = 0$ in a neighborhood \mathcal{V} of $\{(\lambda_1, t, 0, 0) : t \in [0, M]\} \cup \{(a, 0, 0, 0) : a \in [\lambda_1, \lambda_2]\}$. Let $\delta > 0$ be small enough so that \mathcal{I}_ϕ is contained in \mathcal{N} . For each $(a, t) \in \mathcal{I}_\phi$, $(a, t\phi + y_\phi(a, t), c_\phi(a, t))$ is a solution of (1).

Let $\lambda_1 < a < \lambda_1 + \delta$ and $u_\dagger(a)$ be the stable solution in Section 2. Suppose $(a, t_\dagger(a), u_\dagger(a) - t_\dagger(a)\phi, 0) \in \mathcal{V}$. Since $(a, t_\dagger(a), y_\phi(a, t_\dagger(a)), c_\phi(a, t_\dagger(a))) \in \mathcal{V}$ and there is only one solution in \mathcal{V} corresponding to each pair (a, t) , we have $y_\phi(a, t_\dagger(a)) = u_\dagger(a) - t_\dagger(a)\phi$, or $u_\phi(a, t_\dagger(a)) := t_\dagger(a)\phi + y_\phi(a, t_\dagger(a)) = u_\dagger(a)$. Now $(a, t_\dagger(a), u_\dagger(a) - t_\dagger(a)\phi, 0)$ will belong to \mathcal{V} for a close to λ_1 and both $u_\dagger(a)$ and $u_\phi(a, t_\dagger(a))$ are continuous. So $u_\phi(a, t_\dagger(a)) = u_\dagger(a)$ and $c_\phi(a, t_\dagger(a)) = 0$ for $\lambda_1 < a < \lambda_1 + \delta$.

We have $\frac{\partial u_\phi}{\partial t}(\lambda_1, t) = \phi$, for all $t \in [0, M]$, and the function u_ϕ is C^1 . By reducing δ if necessary, we may assume $\frac{\partial u_\phi}{\partial t}$ is a strictly positive function at

each point of \mathcal{I}_ϕ . So let $\lambda_1 \leq a < \lambda_1 + \delta$ and $0 \leq t_1 < t_2 \leq t_+(a)$. Then $u_\phi(a, t_1) < u_\phi(a, t_2)$, i.e. for fixed a , u_ϕ is strictly increasing along the curve of solutions joining zero to the stable solution.

We know $c_\phi(\lambda_1, t) = 0$ for $t \leq M$, and $c_\phi(a, 0) = 0$. We also know $c_\phi(a, t_+(a)) = 0$. Differentiating $\tilde{g}(a, t, y_\phi(a, t), c_\phi(a, t)) = 0$ with respect to t , at $(a, 0)$,

$$\Delta z + az - \gamma h = -(a - \lambda_1)\phi.$$

Here $z = \frac{\partial y_\phi}{\partial t}$ and $\gamma = \frac{\partial c_\phi}{\partial t}$ at $(a, 0)$. This implies

$$\frac{\partial c_\phi}{\partial t}(a, 0) = \frac{\int \phi^2}{\int h\phi} (a - \lambda_1). \quad (22)$$

If we fix $\lambda_1 < a < \lambda_1 + \delta$ and start increasing t from zero, (22) shows that initially c_ϕ increases. We denote by $(\mu_\phi(a, t), w_\phi(a, t))$ the first eigenpair of the linearized problem at $u_\phi(a, t)$. Another application of the Implicit Function Theorem shows this eigenpair has a C^1 dependence on (a, t) . Differentiating (1) with respect to t and using the definition of (μ_ϕ, w_ϕ) ,

$$\begin{cases} \Delta v + av - f'(u_\phi)v &= \gamma h, \\ \Delta w_\phi + aw_\phi - f'(u_\phi)w_\phi &= -\mu_\phi w_\phi, \end{cases}$$

where $v = \frac{\partial u_\phi}{\partial t}$ and γ is as before. This implies

$$-\mu_\phi \int \frac{\partial u_\phi}{\partial t} w_\phi = \frac{\partial c_\phi}{\partial t} \int h w_\phi. \quad (23)$$

By (23), μ_ϕ is negative as long as $\frac{\partial c_\phi}{\partial t} > 0$. On the other hand the equality $c_\phi(a, t_+(a)) = 0$, implies $\frac{\partial c_\phi}{\partial t}(a, \bar{t}(a)) = 0$ for some $0 < \bar{t}(a) < t_+(a)$. If $\frac{\partial c_\phi}{\partial t}(a, \bar{t}(a))$ vanishes, $(a, u_\phi(a, \bar{t}(a)), c_\phi(a, \bar{t}(a)))$ is a degenerate solution with Morse index equal to zero. By the uniqueness statement of Theorem 3.1, it belongs to \mathcal{D}_* and $\bar{t}(a) = t_*(a)$, where $t_*(a)$ is as in (14). Now $\mu_\phi(a, t) < 0$ for $0 \leq t < t_*(a)$, and $\mu_\phi(a, t) > 0$ for $t_*(a) < t \leq t_+(a)$ by Lemma 4.3, otherwise we would obtain more than one solution on \mathcal{D}_* for a fixed value of a . From (23), $\frac{\partial c_\phi}{\partial t}(a, t) > 0$ for $0 \leq t < t_*(a)$, and $\frac{\partial c_\phi}{\partial t}(a, t) < 0$ for $t_*(a) < t \leq t_+(a)$. Therefore, $c_\phi(a, t)$ increases strictly for t in $[0, t_*(a)]$ and decreases strictly for t in $[t_*(a), t_+(a)]$.

To see that there are no other solutions of (1) for $\lambda_1 \leq a < \lambda_1 + \delta$ and $c \geq 0$, we may argue by contradiction. If there were we could follow them using the parameter c until $c_*(a)$. A contradiction would result from Lemma 4.3 (see the proof of Theorem 1.1 ahead). In alternative we could just appeal to Theorem 1.1. \square

Proof of Remark 5.2. Going back to the proof of Theorem 3.1, reducing σ if necessary, the curve parametrized by $t \mapsto (a_*(t), t\phi + y_*(t), c_*(t))$ for $t \in]M, M + \sigma[$ may be parametrized in the form $a \mapsto (a, u_*(a), c_*(a))$, i.e. may be parametrized in terms of a . Under the weaker condition **(b)''** the curve \mathcal{D}_* may afterwards turn back (so that it can no longer be parametrized in terms of a). But, arguing as in Lemma 7.4 ahead, afterwards it will stay away from $\{\lambda_1\} \times \mathcal{H} \times \mathbb{R}$. So we may argue as in Section 7 to see $t \mapsto c_\phi(a, t)$ first increases and then decreases, with no other oscillations, provided δ is chosen sufficiently small. \square

For each fixed a , with $\lambda_1 < a < \lambda_2$, a complete description of the solutions of (1) is given in Theorem 1.1.

Proof of Theorem 1.1. Fix $\lambda_1 < a < \lambda_2$. The Morse index of any solution (a, u, c) of (1) is less than or equal to one and solutions with Morse index equal to one are nondegenerate. Indeed, suppose $v \in \mathcal{H}$ is an eigenfunction for the linearized equation at u , with $L^2(\Omega)$ norm equal to one, and μ is the corresponding eigenvalue. Then $Q_a(v) = \mu$, where Q_a is as in (21). The term $\int f'(u)v^2$ is nonnegative, so if Q_a was nonpositive on a two dimensional space, then $\int (|\nabla v|^2 - av^2)$ would be nonpositive on that space, which is impossible, as $a < \lambda_2$.

We start at $(c, u) = (0, u_+(a))$, where $u_+(a)$ is the stable solution of Section 2, and we start at $(c, u) = (0, 0)$, a nondegenerate solution with Morse index equal to one. From Section 3, there exists only one degenerate solution and that solution corresponds to a positive value of c . Arguing as in the proof of Theorem 4.1, we may use the Implicit Function Theorem to follow the solutions, as c increases, until they become degenerate. From (18), this will have to happen, and for a positive value of c , the value $c = c_*(a)$, where $c_*(a)$ is as in Theorem 3.1. And we may decrease c from zero and follow the solutions for all negative values of c .

The upper bound for c in inequality (18), and the analysis in Lemma 4.3, of the behavior of solutions around a degenerate solution, imply there can be no other branch of solutions besides the one that goes through $(0, u_+(a))$ and $(0, 0)$. \square

6 Global bifurcation at λ_2

In addition to the previous hypotheses (f satisfies **(i)-(iv)** and h satisfies **(a)-(b)**), henceforth we assume the domain Ω is such that **(α)** holds and the function h satisfies **(c)**. Recall the definition of β in (2).

Lemma 6.1 (Morse indices and nondegeneracy at λ_2). *For $a = \lambda_2$ solutions of (1) have Morse index less than or equal to one. All solutions with Morse index equal to one are nondegenerate, except $(\lambda_2, u, c) = (\lambda_2, t\psi, 0)$ for $t \in [-\frac{M}{\beta}, M]$.*

Proof. Let (λ_2, u, c) be a degenerate solution with Morse index equal to one. The quadratic form Q_{λ_2} in (21) is nonpositive on a two dimensional space $\mathcal{E} \subset \mathcal{H}$. The quadratic form $Q : H_0^1(\Omega) \rightarrow \mathbb{R}$, defined by $Q(v) := \int (|\nabla v|^2 - \lambda_2 v^2)$, is also nonpositive on $\mathcal{E} \subset H_0^1(\Omega)$.

We claim \mathcal{E} contains ψ . To prove the claim let v_1 and v_2 be two linearly independent functions in \mathcal{E} . We write

$$\begin{aligned} v_1 &= \eta_1 \phi + \eta_2 \psi + r_1, \\ v_2 &= \rho_1 \phi + \rho_2 \psi + r_2, \end{aligned}$$

with r_1, r_2 orthogonal in $L^2(\Omega)$ to ϕ and ψ . If $\eta_1 = 0$, then $r_1 = 0$ because Q is nonpositive on \mathcal{E} . In this case the claim is proved. Similarly if $\rho_1 = 0$. Suppose now η_1 and ρ_1 are both different from zero. The function $v := v_2 - \frac{\rho_1}{\eta_1} v_1$ is of the form $\chi \psi + r$, with r orthogonal to ϕ and ψ . So $r = 0$. As v cannot be zero, $\chi \neq 0$. The claim is proved.

As $\psi \in \mathcal{E}$ and $\psi^2 \neq 0$ a.e. in Ω , it follows that $f'(u) \equiv 0$. So $u \leq M$. Thus

$$\Delta u + \lambda_2 u - ch = 0.$$

Hypothesis (c) implies $c = 0$. We obtain $u = t\psi$ and, since $u \leq M$, $t \in [-\frac{M}{\beta}, M]$. On the other hand, the solutions $(a, u, c) = (\lambda_2, t\psi, 0)$ with $t \in [-\frac{M}{\beta}, M]$ are degenerate and have Morse index equal to one. \square

Recall the definition of \mathcal{S} in (4).

Lemma 6.2 ($\mathcal{L}^{b\sharp}$, solutions around zero at λ_2). *There exists $\delta > 0$ and C^1 functions $y^{b\sharp} : J \rightarrow \mathcal{S}$ and $c^{b\sharp} : J \rightarrow \mathbb{R}$, where $J =]-\frac{M}{\beta} - \delta, M + \delta[$, such that the map $t \mapsto (\lambda_2, t\psi + y^{b\sharp}(t), c^{b\sharp}(t))$, defined in J , with $y^{b\sharp}(t) = 0$ and $c^{b\sharp}(t) = 0$ for $t \in [-\frac{M}{\beta}, M]$, parametrizes a curve $\mathcal{L}^{b\sharp}$ of solutions of (1). There exists a neighborhood of $\mathcal{L}^{b\sharp}$ in $\{\lambda_2\} \times \mathcal{H} \times \mathbb{R}$ such that the solutions of (1) with $a = \lambda_2$ in this neighborhood lie on $\mathcal{L}^{b\sharp}$.*

Sketch of the proof. Consider the function $\hat{g} : \mathbb{R} \times \mathcal{S} \times \mathbb{R} \rightarrow L^p(\Omega)$, defined by

$$\hat{g}(t, y, c) = \Delta(t\psi + y) + \lambda_2(t\psi + y) - f(t\psi + y) - ch.$$

Let $t_0 \in [-\frac{M}{\beta}, M]$. We know $\hat{g}(t_0, 0, 0) = 0$. The lemma follows from the Implicit Function Theorem. \square

Lemma 6.3 (Solutions at λ_2 for $c = 0$). *Suppose u is a solution of (5) with $a = \lambda_2$. Then either u is a stable solution, or u is a degenerate solution with Morse index equal to one.*

Proof. If u is less than or equal to M , for then u solves $\Delta u + \lambda_2 u = 0$. Thus $u = t\psi$ with $t \in [-\frac{M}{\beta}, M]$ and u is degenerate with Morse index equal to one. It remains to consider the case $\max_{\Omega} u > M$.

We multiply (5) first by u^- and then by u^+ to obtain

$$\int (|\nabla u^-|^2 - \lambda_2 |u^-|^2) = 0$$

and

$$\int (|\nabla u^+|^2 - \lambda_2 |u^+|^2) = - \int f(u)u \leq 0. \quad (24)$$

Thus

$$\int |\nabla u^-|^2 = \lambda_2 \int |u^-|^2 \quad \text{and} \quad \int |\nabla u^+|^2 \leq \lambda_2 \int |u^+|^2.$$

Since u^- and u^+ are orthogonal in $L^2(\Omega)$ and in $H_0^1(\Omega)$, the Dirichlet quotient,

$$\frac{\int |\nabla v|^2}{\int v^2},$$

is less than or equal to λ_2 on the space $\mathcal{E} \subset H_0^1(\Omega)$ spanned by u^- and u^+ (with zero removed). If $u^- \equiv 0$, then u is a nonnegative solution of (5) with maximum strictly greater than M . By Lemma 2.4, u is stable.

Now we consider the case when $u^- \not\equiv 0$. Then the space \mathcal{E} is two dimensional. As in the proof of Lemma 6.1, \mathcal{E} contains ψ . Let η_1 and η_2 be such that

$$\psi = \eta_1 u^+ - \eta_2 u^-.$$

The equality

$$\int (|\nabla \psi|^2 - \lambda_2 |\psi|^2) = 0$$

implies

$$\eta_1^2 \int (|\nabla u^+|^2 - \lambda_2 |u^+|^2) = -\eta_2^2 \int (|\nabla u^-|^2 - \lambda_2 |u^-|^2) = 0.$$

Clearly $\eta_1 \neq 0$, so

$$\int (|\nabla u^+|^2 - \lambda_2 |u^+|^2) = 0.$$

From (24),

$$\int f(u)u = 0.$$

Therefore u is less than or equal to M . We have reached a contradiction. The lemma is proved. \square

We are now in a position to give the

Proof of Theorem 1.2. Clearly $(y^{b\sharp})'(-\frac{M}{\beta}) = 0$ and $(y^{b\sharp})'(M) = 0$. Let δ be as in Lemma 6.2. Reducing δ if necessary, we may assume $\max_{\Omega}(t\psi + y^{b\sharp}(t)) > M$ for $t \in]M, M + \delta[$, and $\max_{\Omega}(t\psi + y^{b\sharp}(t)) > M$ for $t \in]-\frac{M}{\beta} - \delta, M[$. By Lemma 6.1, and further reducing δ if necessary to guarantee the Morse index remains equal to one, we may assume that for $t \in]-\frac{M}{\beta} - \delta, M[\cup]M, M + \delta[$ the solutions $(\lambda_2, t\psi + y^{b\sharp}(t), c^{b\sharp}(t))$ are nondegenerate with Morse index equal to one. Consequently, in a neighborhood of each one of these solutions, the curve $\mathcal{L}^{b\sharp}$ may be written in the form $(\lambda_2, u^{b\sharp}(c), c)$. Therefore the restriction of $c^{b\sharp}$ to $] -\frac{M}{\beta} - \delta, M[$ must be monotone, and the restriction of $c^{b\sharp}$ to $]M, M + \delta[$ must be monotone.

We start at $(c, u) = (0, u_{\dagger}(\lambda_2))$, where $u_{\dagger}(\lambda_2)$ is the stable solution of Section 2. We may use the Implicit Function Theorem to follow the solutions as c increases, until $c = c_*(\lambda_2)$ where $c_*(\lambda_2)$ is as in Theorem 3.1. And we may decrease c from zero and follow the solutions for all negative values of c . The analysis in Lemma 4.3, of the behavior of solutions around a degenerate solution, shows that we may decrease c from $c_*(\lambda_2)$ and follow a branch $(c, u^{\sharp}(c))$ of solutions of Morse index equal to one. Lemma 6.1 implies we may follow this branch until $c = 0$. By Lemma 6.3 when c reaches zero the solution becomes degenerate. When this happens Lemma 6.1 says it must be of the form $t\psi$, with $t \in [-\frac{M}{\beta}, M]$. And Lemma 6.2 says the branch must connect to the branch of solutions described in that lemma. Without loss of generality we may assume $\lim_{c \searrow 0} u^{\sharp}(c) = M\psi$. [(Otherwise we rename ψ to $-\psi$). But, in fact, we will see in the next section that this will be the case if we choose ψ so that $\int h\psi < 0$]. So the restriction of $c^{b\sharp}$ to $]M, M + \delta[$ is increasing. The restriction of $c^{b\sharp}$ to $] -\frac{M}{\beta} - \delta, M[$ must also be increasing so that $c^{b\sharp}$ is negative in this interval. Otherwise we could increase c to $c = c_*(\lambda_2)$ and obtain a contradiction. Indeed, there would be another branch of solutions arriving at the degenerate solution with Morse index equal to zero on the curve \mathcal{D}_* of Section 3, and this would violate Lemma 4.3. Since $c^{b\sharp}$ is negative in the interval $] -\frac{M}{\beta} - \delta, M[$, Lemma 6.1 implies we may use the parameter c to follow this branch, which we call $(c, u^b(c))$, for negative values of c . In fact, with the aid of Remark 3.3, we

may follow the branch $(c, u^b(c))$ for all negative values of c . Clearly, there can be no other branches of solutions besides the one just described. \square

Proof of Remark 1.3. Using the notations of Lemma 6.2, let $u^{b\sharp}(t) = t\psi + y^{b\sharp}(t)$. One easily computes

$$\frac{dc^{b\sharp}}{dt}(0) = 0 \quad \text{and} \quad \frac{dy^{b\sharp}}{dt}(0) = 0.$$

By the chain rule,

$$\frac{du^{b\sharp}}{dc^{b\sharp}}(0) = \frac{du^{b\sharp}}{dt}(0) \frac{dt}{dc^{b\sharp}}(0) = \psi \times \frac{1}{0} = \infty.$$

\square

7 Bifurcation in a right neighborhood of λ_2

In this and the next sections, we will determine the bifurcation curves of (1) when the value of a is in a certain right neighborhood of λ_2 , which will not include λ_3 . By arguing as in Section 5, in this situation any solution of (1) will have Morse index less than or equal to two, and if it has Morse index equal to two then it is nondegenerate.

Regarding solutions of (5), and analogously to Lemma 2.3, we have

Proposition 7.1 (\mathcal{C}_\dagger , solutions of (5) bifurcating from $(\lambda_2, 0)$). *Suppose f satisfies (i)-(iv) and (α) holds. Suppose also $M > 0$. There exists $\delta > 0$ and C^1 functions $a_\dagger : J \rightarrow \mathbb{R}$ and $y_\dagger : J \rightarrow \mathcal{S}$, where $J =]-\frac{M}{\beta} - \delta, M + \delta[$, such that the map $t \mapsto (a_\dagger(t), t\psi + y_\dagger(t))$, defined in J , with $a_\dagger(t) = \lambda_2$ and $y_\dagger(t) = 0$ for $t \in [-\frac{M}{\beta}, M]$, parametrizes a curve \mathcal{C}_\dagger of solutions of (5). There exists a neighborhood of $\mathcal{C}_\dagger \setminus \{(\lambda_2, 0)\}$ in $\mathbb{R} \times \mathcal{H}$ such that the solutions of (5) in this neighborhood lie on \mathcal{C}_\dagger .*

The proof is similar to the one of Lemma 2.3.

We again refer to the classical paper [9, Theorem 1.7] to assert $(\lambda_2, 0)$ is a bifurcation point. The statement of Proposition 7.1 also holds when $M = 0$. In both cases, $M > 0$ and $M = 0$, in a neighborhood of $(\lambda_2, 0)$ the solutions of (5) lie on $\mathcal{A} \cup \mathcal{C}_\dagger$. Once more \mathcal{A} is the line parametrized by $a \mapsto (a, 0)$.

From Section 5 we know the only solutions of (5) for $\lambda_1 < a < \lambda_2$ are zero and the stable solution. From Section 6 we know the only solutions of (5) for $a = \lambda_2$ are the stable solution and $t\psi$ with $t \in [-\frac{M}{\beta}, M]$. Recalling that these (this in the case $M = 0$) last solutions are (is) degenerate with Morse index equal to one, it follows that $a_\dagger(t) > \lambda_2$ if $t \in]-\frac{M}{\beta} - \delta, -\frac{M}{\beta}[\cup]M, M + \delta[$.

Lemma 7.2 (\mathcal{D}_ζ , degenerate solutions with Morse index equal to one). *There exists $\sigma > 0$ and C^1 functions $a_\zeta : J \rightarrow \mathbb{R}$, $y_\zeta : J \rightarrow \mathcal{S}$, $c_\zeta : J \rightarrow \mathbb{R}$ and $\zeta_\zeta : J \rightarrow \{\zeta \in \mathcal{H} : \int \zeta^2 = \int \psi^2\}$, where $J =]-\frac{M}{\beta} - \sigma, M + \sigma[$, such that the map $t \mapsto (a_\zeta(t), u_\zeta(t), c_\zeta(t))$, defined in J , where $u_\zeta(t) = t\psi + y_\zeta(t)$, with $a_\zeta(t) = \lambda_2$, $y_\zeta(t) = 0$, $c_\zeta(t) = 0$, $\zeta_\zeta(t) = \psi$ for $t \in [-\frac{M}{\beta}, M]$, parametrizes a curve \mathcal{D}_ζ of degenerate solutions of (1) with Morse index equal to one. There exists a neighborhood \mathcal{O} of \mathcal{D}_ζ in $\mathbb{R} \times \mathcal{H} \times \mathbb{R}$ such that the degenerate solutions of the equation in \mathcal{O} lie on \mathcal{D}_ζ . The degenerate directions are given by ζ_ζ .*

The proof is similar to the one of Lemma 3.2.

From Section 5 we know the solutions of (1) for $\lambda_1 < a < \lambda_2$ are never degenerate with Morse index equal to one. From Section 6 we know the only degenerate solutions with Morse index equal to one of (1) for $a = \lambda_2$ are $t\psi$ with $t \in [-\frac{M}{\beta}, M]$. It follows that

$$a_\zeta(t) > \lambda_2 \quad \text{if} \quad t \in]-\frac{M}{\beta} - \sigma, -\frac{M}{\beta}[\cup]M, M + \sigma[.$$

Lemma 7.3 (Solutions around $\{(\lambda_2, t\psi, 0) : t \in [-\frac{M}{\beta}, M]\}$). *There exists a neighborhood $\check{\mathcal{V}}$ of $\{(\lambda_2, t\psi, 0) : t \in [-\frac{M}{\beta}, M]\}$ in $\mathbb{R} \times \mathcal{H} \times \mathbb{R}$, ε and $\tilde{\varepsilon} > 0$ such that solutions (a, u, c) of (1) in $\check{\mathcal{V}}$ can be parametrized in the chart*

$$\mathcal{I}_\psi :=]\lambda_2 - \varepsilon, \lambda_2 + \varepsilon[\times]-\frac{M}{\beta} - \tilde{\varepsilon}, M + \tilde{\varepsilon}[.$$

by

$$(a, t) \mapsto (a, u_\psi(a, t), c_\psi(a, t)) = (a, t\psi + y_\psi(a, t), c_\psi(a, t))$$

Here $y_\psi : \mathcal{I}_\psi \rightarrow \mathcal{S}$ and $c_\psi : \mathcal{I}_\psi \rightarrow \mathbb{R}$ are C^1 functions. We have $u_\psi(\lambda_2, t) = t\psi$ and $c_\psi(\lambda_2, t) = 0$, for $t \in [-\frac{M}{\beta}, M]$.

The proof is similar to the argument in the first paragraph of the proof of Theorem 5.1.

We choose the value $\tilde{\varepsilon}$ such that $a_\zeta(t) < \lambda_2 + \varepsilon$ for $t \in]-\frac{M}{\beta} - \tilde{\varepsilon}, M + \tilde{\varepsilon}[$. We may assume $\check{\mathcal{V}} \subset \mathcal{O}$ and $\tilde{\varepsilon}$ is smaller than σ . Here \mathcal{O} and σ are as in Lemma 7.2. We also assume ε and $\tilde{\varepsilon}$ are small enough so that the solutions of (1) in the closure of $\check{\mathcal{V}}$ have Morse index at least equal to one.

Lemma 7.4 (No degenerate solutions with Morse index equal to one outside $\check{\mathcal{V}}$ for c bounded below). *Let $L > 0$. There exists $\hat{\delta}_L > 0$ such that degenerate solutions (a, u, c) of (1), with Morse index equal to one, $a < \lambda_2 + \hat{\delta}_L$ and $c \geq -L$, lie in the set $\check{\mathcal{V}}$ of Lemma 7.3.*

In the next section we will prove that actually one can choose $\hat{\delta}$ independently of L .

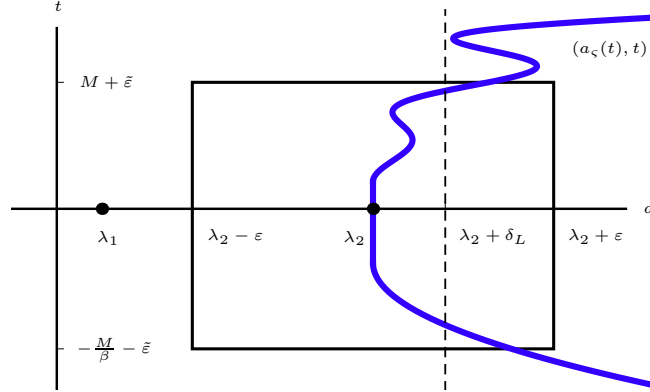


Figure 7: The chart of Lemma 7.3 and the choice of the value of δ_L .

Proof of Lemma 7.4. We argue by contradiction. Suppose (a_n, u_n, c_n) is a sequence of degenerate solutions of (1), with Morse index equal to one, $a_n \leq \lambda_2 + o(1)$ and $c_n \geq -L$ lying outside $\check{\mathcal{V}}$. Of course (c_n) is bounded above (recall (18) and the solutions are bounded above by K in (20)). We may assume $c_n \rightarrow c$. From Sections 5 and 6, we may also assume $a_n \searrow \lambda_2$. From Remark 3.3, (u_n) is uniformly bounded. By [11, Lemma 9.17] the norms $\|u_n\|_{\mathcal{H}}$ are uniformly bounded. Thus (u_n) has a strongly convergent subsequence in $L^p(\Omega)$. Subtracting equations (1) for u_n and u_m and again using [11, Lemma 9.17], (u_n) has a subsequence which is strongly convergent, say to u , in \mathcal{H} . One easily sees that (λ_2, u, c) is a degenerate solution of (1) with Morse index equal to one. But (λ_2, u, c) belongs to the closure of the complement of $\check{\mathcal{V}}$. This contradicts Lemma 6.1 and finishes the proof. \square

We define

$$\delta_L = \min\{\hat{\delta}_L, a_\varsigma(M + \tilde{\varepsilon}) - \lambda_2, a_\varsigma(-\frac{M}{\beta} - \tilde{\varepsilon}) - \lambda_2\} \quad (25)$$

(see Figure 7). Here a_ς is as in Lemma 7.2 and $\tilde{\varepsilon}$ is as in Lemma 7.3.

We will prove Theorem 1.4 in the next section. For now we prove the following weaker statement.

Proposition 7.5 (Bifurcation in a right neighborhood of λ_2 for c bounded below). *Suppose f satisfies (i)-(iv), (α) holds and h satisfies (a)-(c). Without loss of generality, suppose (3) is true. Let $L > 0$ be large and δ_L be as in (25). Fix $\lambda_2 < a < \lambda_2 + \delta_L$. The set of solutions of (1) with $c > -L$ is a manifold as described in Theorem 1.4, with u^* defined in $] -L, c_*[$ and u^b defined in $] -L, c_b[$.*

Proof. We start at $(a, u, c) = (a, 0, 0)$. In the (a, t) coordinates of Lemma 7.3 this solution may also be written as $(a, 0)$ as $(a, 0\psi + y_\psi(a, 0), c_\psi(a, 0)) = (a, 0, 0)$. Analogously to (22),

$$\frac{\partial c_\psi}{\partial t}(a, 0) = \frac{\int \psi^2}{\int h\psi} (a - \lambda_2).$$

This shows

$$\frac{\partial c_\psi}{\partial t}(a, 0) > 0 \text{ for } a < \lambda_2 \quad \text{and} \quad \frac{\partial c_\psi}{\partial t}(a, 0) < 0 \text{ for } a > \lambda_2. \quad (26)$$

Let (a, u, c) be a solution with coordinates (a, t) . We claim that if $a < a_\varsigma(t)$ then $\frac{\partial c_\psi}{\partial t}(a, t) > 0$, and if $a > a_\varsigma(t)$ then $\frac{\partial c_\psi}{\partial t}(a, t) < 0$. Indeed, first we observe that if $a \neq a_\varsigma(t)$ then $\frac{\partial c_\psi}{\partial t}(a, t) \neq 0$, because if $\frac{\partial c_\psi}{\partial t}(a, t) = 0$, then, differentiating (1) with respect to t , we conclude that the solution with coordinates (a, t) is degenerate. But we know from Lemma 7.2 the only degenerate solutions of (1) in $\check{\mathcal{V}}$ lie on \mathcal{D}_ς . The curve parametrized by $t \mapsto (a_\varsigma(t), t)$ divides the rectangle \mathcal{I}_ψ in two components, $\mathcal{O}_1 := \{(a, t) \in \mathcal{I}_\psi : a < a_\varsigma(t)\}$ and $\mathcal{O}_2 := \{(a, t) \in \mathcal{I}_\psi : a > a_\varsigma(t)\}$. The segment $(a, 0)$ with $a < \lambda_2$ is contained in the first component \mathcal{O}_1 , and the segment $(a, 0)$ with $a > \lambda_2$ is contained in the second component \mathcal{O}_2 . Using inequalities (26) and the continuity of $\frac{\partial c_\psi}{\partial t}$, $\frac{\partial c_\psi}{\partial t}(a, t) > 0$ in \mathcal{O}_1 and $\frac{\partial c_\psi}{\partial t}(a, t) < 0$ in \mathcal{O}_2 .

Since a is smaller than $\lambda_2 + \delta$, and δ is smaller than $a_\varsigma(M + \tilde{\varepsilon}) - \lambda_2$, we have $a < a_\varsigma(M + \tilde{\varepsilon})$, which means $(a, M + \tilde{\varepsilon}) \in \mathcal{O}_1$. Similarly, $a < a_\varsigma(-\frac{M}{\beta} - \tilde{\varepsilon})$ so that also $(a, -\frac{M}{\beta} - \tilde{\varepsilon}) \in \mathcal{O}_1$. Thus

$$\frac{\partial c_\psi}{\partial t}(a, M + \tilde{\varepsilon}) > 0, \quad (27)$$

$$\frac{\partial c_\psi}{\partial t}(a, -\frac{M}{\beta} - \tilde{\varepsilon}) > 0. \quad (28)$$

As mentioned above, the parameter a is fixed. We vary t in $[-\frac{M}{\beta} - \tilde{\varepsilon}, M + \tilde{\varepsilon}]$ to follow the solutions of (1) in $\check{\mathcal{V}}$. When we arrive at the solution with coordinates $(a, M + \tilde{\varepsilon})$, (27) shows we may follow the solutions out of $\check{\mathcal{V}}$ by changing coordinates, using c as a parameter and increasing it. On the opposite side, when we arrive at the solution with coordinates $(a, -\frac{M}{\beta} - \tilde{\varepsilon})$, (28) shows we may follow the solutions out of $\check{\mathcal{V}}$ by changing coordinates, using c as a parameter and decreasing it. Indeed, c is increasing as one enters $\check{\mathcal{V}}$ using t as an increasing parameter.

We observe that again by continuity and because the only degenerate solutions of (1) in $\check{\mathcal{V}}$ lie on \mathcal{D}_ς , the Morse index of solutions in \mathcal{O}_1 is equal to

one and the Morse index of solutions in \mathcal{O}_2 is equal to two. So the solutions with coordinates $(a, M + \tilde{\varepsilon})$ and $(a, -\frac{M}{\beta} - \tilde{\varepsilon})$ are nondegenerate and have Morse index equal to one.

When we arrive at the solution with coordinates $(a, M + \tilde{\varepsilon})$, and follow the solutions out of $\check{\mathcal{V}}$ by increasing c , Lemma 7.4 says if we find a degenerate solution it will have to have Morse index equal to zero. Similarly, when we arrive at the solution with coordinates $(a, -\frac{M}{\beta} - \tilde{\varepsilon})$, and follow the solutions out of $\check{\mathcal{V}}$ by decreasing c , Lemma 7.4 says if we find a degenerate solution it will have to have Morse index equal to zero. But we know these lie on \mathcal{D}_* . So one can finish by arguing as in the proof of Theorem 1.1.

The value ε_{\sharp} in the statement of Theorem 1.4 is the maximum value of $t \in]0, \tilde{\varepsilon}[$ such that $a = a_{\zeta}(M + t)$. The value ε_{\flat} is the maximum value of $t \in]0, \tilde{\varepsilon}[$ such that $a = a_{\zeta}(-\frac{M}{\beta} - t)$. Also, $(c_{\flat}, u_{\flat}) = (c_{\psi}(a, -\frac{M}{\beta} - \varepsilon_{\flat}), u_{\psi}(a, -\frac{M}{\beta} - \varepsilon_{\flat}))$ and $(c_{\sharp}, u_{\sharp}) = (c_{\psi}(a, M + \varepsilon_{\sharp}), u_{\psi}(a, M + \varepsilon_{\sharp}))$.

It is clear that if $|c|$ is sufficiently small, then (1) has at least four solutions. \square

Proof of Remark 1.5. This is the assertion that if $\frac{\partial c_{\psi}}{\partial t}(a, t) > 0$, then $(a, t) \in \mathcal{O}_1$ and so the solution with coordinates (a, t) is nondegenerate with Morse index equal to one; if $\frac{\partial c_{\psi}}{\partial t}(a, t) < 0$, then $(a, t) \in \mathcal{O}_2$ and so the solution with coordinates (a, t) is nondegenerate with Morse index equal to two; if $\frac{\partial c_{\psi}}{\partial t}(a, t) = 0$, then $a = a_{\zeta}(t)$ and so the solution with coordinates (a, t) is degenerate with Morse index equal to one.

In alternative, at the expense of maybe another reduction in \mathcal{I}_{ψ} , we could argue using the analogue of (23), namely

$$-\mu_{\psi} \int \frac{\partial u_{\psi}}{\partial t} \zeta_{\psi} = \frac{\partial c_{\psi}}{\partial t} \int h \zeta_{\psi}.$$

Here $(\mu_{\psi}, \zeta_{\psi})$ is the second eigenpair associated with linearized problem for the solution (a, u_{ψ}, c_{ψ}) , obtained from the Implicit Function Theorem and with $(\mu_{\psi}(\lambda_2, t), \zeta_{\psi}(\lambda_2, t)) = (0, \psi)$ for $t \in [-\frac{M}{\beta}, M]$. \square

8 Global bifurcation in a right neighborhood of λ_2

In this section we prove Theorem 1.4. We assume f satisfies (i)-(iv), (α) holds and h satisfies (a), (b)', (c). The argument rests on the results above and on

Lemma 8.1 (No degenerate solutions with Morse index equal to one at infinity for $a < \lambda_2 + \delta$). *There exists $\delta_0 > 0$ and $c_0 < 0$ such that degenerate solutions (a, u, c) of (1) with Morse index equal to one and $a < \lambda_2 + \delta_0$ satisfy $c \geq c_0$.*

Proof. We argue by contradiction. Let $((a_n, u_n, c_n))$ be a sequence of degenerate solutions of (1) with Morse index equal to one, $a_n < \lambda_2 + \frac{1}{n}$ and $c_n \rightarrow -\infty$. We assume $c_n < 0$ for all n . As we saw in the proofs of Theorem 1.1 and Lemma 6.1, $a_n > \lambda_2$. So $a_n \rightarrow \lambda_2$.

Let (μ_n, w_n) be the first eigenpair associated with linearized problem for the solution (a_n, u_n, c_n) , with $w_n > 0$ and $\int w_n^2 = \int \phi^2$. We have

$$\Delta w_n + a_n w_n - f'(u_n) w_n = -\mu_n w_n \quad (29)$$

and $\lambda_1 - a_n < \mu_n < 0$. Multiplying both sides of (29) by w_n and integrating it follows that (w_n) is bounded in $H_0^1(\Omega)$. We may assume $w_n \rightarrow w$ in $H_0^1(\Omega)$, $w_n \rightarrow w$ in $L^2(\Omega)$, and $w_n \rightarrow w$ a.e. in Ω . Clearly $w \geq 0$ and $\int w^2 = \int \phi^2$.

Using (1) and (29),

$$0 \leq \int (f'(u_n) u_n - f(u_n)) w_n = c_n \int h w_n + \mu_n \int u_n w_n,$$

or

$$-c_n \int h w_n \leq -\mu_n \int u_n^- w_n \leq |\mu_n| \|u_n\|_{L^2(\Omega)} \|\phi\|_{L^2(\Omega)}.$$

Now

$$\int h w_n \rightarrow \int h w > 0,$$

as h satisfies **(b)'**. It follows there exists a constant $C > 0$ such that

$$-\frac{c_n}{\|u_n\|_{L^2(\Omega)}} \leq C. \quad (30)$$

Defining

$$v_n = \frac{u_n}{\|u_n\|_{L^2(\Omega)}} \quad \text{and} \quad d_n = \frac{c_n}{\|u_n\|_{L^2(\Omega)}},$$

the new functions satisfy

$$\Delta v_n + a_n v_n - \frac{f(u_n)}{\|u_n\|_{L^2(\Omega)}} - d_n h = 0. \quad (31)$$

Inequality (30) shows we may assume $d_n \rightarrow d$.

Claim 8.2. *There exists $C > 0$ such that $f(u_n) \leq C(-c_n)$.*

Proof of Claim 8.2. Let x_n be a point of maximum of u_n . As $(\Delta u_n)(x_n) \leq 0$

$$a_n u_n(x_n) - f(u_n(x_n)) - c_n h(x_n) \geq 0,$$

or, as $u_n(x_n) > M \geq 0$ (because $a_n > \lambda_2$ and the solutions are degenerate),

$$f(u_n(x_n)) \leq a_n u_n(x_n) - c_n h(x_n), \quad (32)$$

$$\frac{f(u_n(x_n))}{u_n(x_n)} \leq a_n - \frac{c_n}{u_n(x_n)} h(x_n). \quad (33)$$

Suppose $u_n(x_n) \geq -c_n$ for large n . Then the right hand side of (33) is bounded. Hypothesis **(iv)** implies $u_n(x_n)$ is bounded and this contradicts $u_n(x_n) \geq -c_n$ as $c_n \rightarrow -\infty$. Therefore

$$u_n(x_n) \leq -c_n \quad \text{for large } n.$$

Using (32),

$$f(u_n(x_n)) \leq (a_n - h(x_n))(-c_n) \leq C(-c_n).$$

As f is increasing, the claim is proved. \square

We return to the proof of Lemma 8.1. Claim 8.2 and (30) together imply

$$\frac{f(u_n)}{\|u_n\|_{L^2(\Omega)}} \quad \text{is uniformly bounded in } L^\infty(\Omega). \quad (34)$$

We may assume

$$\frac{f(u_n)}{\|u_n\|_{L^2(\Omega)}} \rightharpoonup f_\infty \quad \text{in } L^2(\Omega).$$

Here $f_\infty \geq 0$. Multiplying both sides of (31) by v_n and integrating, (v_n) is bounded in $H_0^1(\Omega)$. We may assume $v_n \rightharpoonup v$ in $H_0^1(\Omega)$, $v_n \rightarrow v$ in $L^2(\Omega)$, and $v_n \rightarrow v$ a.e. in Ω . In fact, [11, Lemma 9.17] and (31) imply $v_n \rightarrow v$ in $C^{1,\alpha}(\bar{\Omega})$. The function v is a weak solution of

$$\Delta v + \lambda_2 v - f_\infty - dh = 0. \quad (35)$$

Suppose $v(x) > 0$ for some $x \in \Omega$. Then, since from (30) the sequence $(\|u_n\|_{L^2(\Omega)})$ is unbounded, using **(iv)**,

$$\frac{f(u_n(x))}{\|u_n\|_{L^2(\Omega)}} = \frac{f(\|u_n\|_{L^2(\Omega)} v_n(x))}{\|u_n\|_{L^2(\Omega)} v_n(x)} v_n(x) \longrightarrow \infty \times v(x) = \infty.$$

This contradicts (34) because on the set $\{x \in \Omega : v(x) > 0\}$ the sequence $\left(\frac{f(u_n)}{\|u_n\|_{L^2(\Omega)}}\right)$ converges pointwise to $+\infty$. Therefore $v \leq 0$ and

$$\sup_{\Omega} v_n \rightarrow 0, \quad (36)$$

as $v_n \rightarrow v$ in $C^{1,\alpha}(\bar{\Omega})$.

Let $(0, \zeta_n)$ be a second eigenpair associated with linearized problem for the solution (a_n, u_n, c_n) , with $\int \zeta_n^2 = \int \psi^2$. We have

$$\Delta \zeta_n + a_n \zeta_n - f'(u_n) \zeta_n = 0. \quad (37)$$

Claim 8.3. *There exists a linear combination of w_n and ζ_n such that*

$$\xi_n := \eta_1 w_n + \eta_2 \zeta_n$$

converges strongly to ψ in $H_0^1(\Omega)$.

Proof of Claim 8.3. From (29),

$$\int |\nabla w_n|^2 \leq a_n \int w_n^2$$

and

$$\int |\nabla w|^2 \leq \lambda_2 \int w^2,$$

where w is as above. On the other hand (37) implies, modulo a subsequence, $\zeta_n \rightharpoonup \zeta$ in $H_0^1(\Omega)$, $\zeta_n \rightarrow \zeta$ in $L^2(\Omega)$, and $\zeta_n \rightarrow \zeta$ a.e. in Ω with

$$\int |\nabla \zeta|^2 \leq \lambda_2 \int \zeta^2.$$

For any linear combination $\hat{\xi}_n := \kappa_1 w_n + \kappa_2 \zeta_n$ we have

$$\Delta \hat{\xi}_n + a_n \hat{\xi}_n - f'(u_n) \hat{\xi}_n = -\mu_n \kappa_1 w_n, \quad (38)$$

and, as $\int w_n \zeta_n = 0$,

$$\int |\nabla \hat{\xi}_n|^2 \leq a_n \int \hat{\xi}_n^2. \quad (39)$$

Hence

$$\int |\nabla(\kappa_1 w + \kappa_2 \zeta)|^2 \leq \lambda_2 \int (\kappa_1 w + \kappa_2 \zeta)^2.$$

Since $\int w \zeta = 0$, w and ζ span a two dimensional space \mathcal{E} . By arguing as in the proof of Lemma 6.1, there exist η_1 and η_2 such that

$$\psi = \eta_1 w + \eta_2 \zeta.$$

We have

$$\eta_1 w_n + \eta_2 \zeta_n \rightharpoonup \eta_1 w + \eta_2 \zeta = \psi \quad \text{in } H_0^1(\Omega).$$

Taking $\kappa_1 = \eta_1$ and $\kappa_2 = \eta_2$ in $\hat{\xi}_n$, passing to the limit in (39), and using the lower semi-continuity of the norm,

$$\int |\nabla \psi|^2 \leq \liminf \int |\nabla(\eta_1 w_n + \eta_2 \zeta_n)|^2 \leq \lambda_2 \int \psi^2.$$

However, $\lambda_2 \int \psi^2 = \int |\nabla \psi|^2$ so

$$\eta_1 w_n + \eta_2 \zeta_n \rightarrow \psi \quad \text{in } H_0^1(\Omega).$$

The claim is proved. \square

We return to the proof of Lemma 8.1. Using (38),

$$\int |\nabla \xi_n|^2 - a_n \int \xi_n^2 + \int f'(u_n) \xi_n^2 = \mu_n \eta_1^2 \int w_n^2. \quad (40)$$

All terms in (40) remain bounded as $n \rightarrow \infty$, except perhaps $\int f'(u_n) \xi_n^2$, which must also therefore remain bounded. Recall $\sup_\Omega u_n > M \geq 0$. Using $f'(u) \geq \frac{f(u)}{u}$, we may estimate this term from below as follows

$$\begin{aligned} \int f'(u_n) \xi_n^2 &\geq \frac{1}{\sup_\Omega v_n} \int \frac{f(u_n)}{\|u_n\|_{L^2(\Omega)}} \xi_n^2 \\ &= \frac{1}{\sup_\Omega v_n} \left[\int \frac{f(u_n)}{\|u_n\|_{L^2(\Omega)}} \psi^2 + \int \frac{f(u_n)}{\|u_n\|_{L^2(\Omega)}} (\xi_n^2 - \psi^2) \right] \\ &\rightarrow \frac{1}{0^+} \left(\int f_\infty \psi^2 + 0 \right). \end{aligned}$$

We have used (36). This proves $f_\infty = 0$ a.e. in Ω , because by the unique continuation principle ψ only vanishes on a set of measure zero.

Returning to (35),

$$\Delta v + \lambda_2 v - dh = 0.$$

Hypothesis (c) implies $d = 0$. Consequently, v must be a multiple of ψ . But $v \leq 0$ so $v = 0$, contradicting $\|v\|_{L^2(\Omega)} = 1$. We have finished the proof of Lemma 8.1. \square

Our last step is the

Proof of Theorem 1.4. Take $L = -c_0$, with c_0 as in Lemma 8.1, let $\hat{\delta}_{-c_0}$ be as in Lemma 7.4 and δ_{-c_0} be as in (25). Finally, let

$$\delta = \min\{\delta_0, \delta_{-c_0}\}.$$

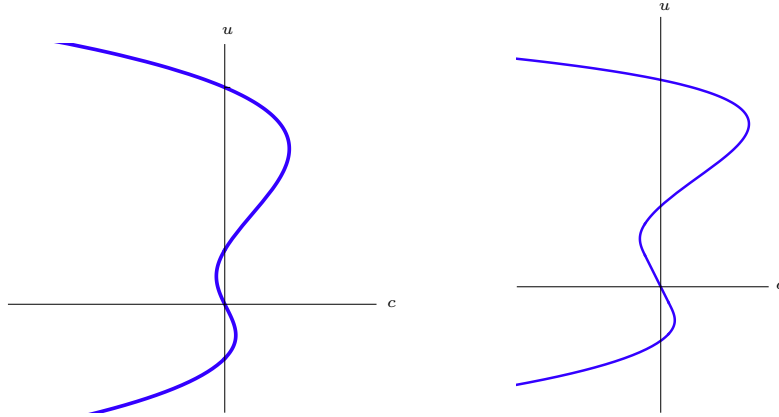


Figure 8: Bifurcation curve for $\lambda_2 < a < \lambda_2 + \delta$. On the left $M = 0$ and on the right $M > 0$.

Here δ_0 is as in Lemma 8.1. Degenerate solutions (a, u, c) of (1) with Morse index equal to one and $a < \lambda_2 + \delta$ lie in the set $\check{\mathcal{V}}$ of Lemma 7.3. This is a consequence of Lemmas 7.4 and 8.1. Using Remark 3.3, we may use the parameter c to follow the branch of solutions in Proposition 7.5 as $c \rightarrow -\infty$. \square

We finish with the simplest bifurcation curves one can obtain for $\lambda_2 < a < \lambda_2 + \delta$. These are illustrated in Figure 8.

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9 Appendix

This appendix is not to appear in the final version of the paper. It contains some proofs omitted in the text.

Proof of Lemma 2.3. Consider the function $g : \mathbb{R}^2 \times \mathcal{R} \rightarrow L^p(\Omega)$, defined by

$$g(a, t, y) = \Delta(t\phi + y) + a(t\phi + y) - f(t\phi + y).$$

Let $t_0 \in]-\infty, 0[\cup]0, M[$. We know $g(\lambda_1, t_0, 0) = 0$. We use the Implicit Function Theorem to show that we may write the solutions of $g(a, t, y) = 0$, in a neighborhood of $(\lambda_1, t_0, 0)$, in the form $(a_{\dagger}(t), t, y_{\dagger}(t))$. Let $(\alpha, z) \in \mathbb{R} \times \mathcal{R}$. The derivative $Dg_{(a,y)}(\alpha, z)$ at $(\lambda_1, t_0, 0)$ is

$$g_a \alpha + g_y z = \alpha t_0 \phi + \Delta z + \lambda_1 z.$$

We check that the derivative is injective. Suppose $\alpha t_0 \phi + \Delta z + \lambda_1 z = 0$. Multiplying both sides of this equation by ϕ and integrating we get $\alpha = 0$. Thus $\Delta z + \lambda_1 z = 0$. Since $z \in \mathcal{R}$ we also get $z = 0$. This proves injectivity. It is easy to check that the derivative is also surjective. So the derivative is a homeomorphism from $\mathbb{R} \times \mathcal{R}$ to $L^p(\Omega)$. \square

Proof of Theorem 4.1. Consider the function $\tilde{H} : \mathcal{H} \times \mathbb{R} \times S \times \mathbb{R} \rightarrow L^p(\Omega) \times L^p(\Omega)$ (S given in (12)), \tilde{H} defined by

$$\tilde{H}(u, c, w, \mu) = (\Delta u + au - f(u) - ch, \Delta w + aw - f'(u)w + \mu w).$$

We may use the Implicit Function Theorem to describe the solutions of $\tilde{H} = 0$ in a neighborhood of a stable solution (u, c, w, μ) . Here μ is the first eigenvalue of the associated linearized problem and w is the corresponding positive eigenfunction on S . Indeed,

$$\begin{aligned} \tilde{H}_u v + \tilde{H}_w \omega + \tilde{H}_\mu \nu &= (\Delta v + av - f'(u)v, \\ &\Delta \omega + a\omega - f'(u)\omega + \mu\omega - f''(u)vw + \nu w). \end{aligned}$$

Consider the system obtained by setting the previous derivative equal to $(0, 0)$. From the first equation we get $v = 0$ because $\mu > 0$. Then, multiplying the second equation by w and integrating by parts, we get $\nu = 0$. Thus ω is a multiple of w . Since ω is orthogonal to w , $\omega = 0$. The derivative is a homeomorphism from $\mathcal{H} \times \mathcal{R}_w \times \mathbb{R}$ to $L^p(\Omega) \times L^p(\Omega)$. So, for each fixed a , the solutions of $\tilde{H} = 0$ in a neighborhood of a stable solution (u, c, w, μ) lie on a C^1 curve parametrized by $c \mapsto (u^*(c), c, w^*(c), \mu^*(c))$. We differentiate both sides of the equations $\tilde{H}(u^*(c), c, w^*(c), \mu^*(c)) = (0, 0)$ with respect to c . We obtain

$$\begin{cases} \Delta v + av - f'(u)v = h, \\ \Delta \omega + a\omega - f'(u)\omega + \mu\omega - f''(u)vw = -\nu w, \end{cases} \quad (41)$$

where $v = (u^*)'(c)$, $\omega = (w^*)'(c)$ and $\nu = (\mu^*)'(c)$. When $\mu > 0$ the first equation and the maximum principle give

$$(u^*)'(c) < 0.$$

From the second equation we get

$$(\mu^*)'(c) = \frac{\int f''(u)vw^2}{\int \phi^2} < 0.$$

By Remark 3.3, we may follow the solution $u^*(c)$ until it becomes degenerate. The solution $u^*(c)$ will have to become degenerate for some value of c . Indeed,

from (1) we obtain

$$c \int h\phi = (a - \lambda_1) \int u\phi - \int f(u)\phi \leq (a - \lambda_1) \int u^+\phi, \quad (42)$$

showing c is bounded above. The solutions $u^*(c)$ cannot be continued for all positive values of c . There must exist c_* such that $\lim_{c \nearrow c_*} \mu^*(c) = 0$. Clearly, the solutions $u^*(c)$ will converge to a solution u_* as $c \nearrow c_*$. By the uniqueness assertion in Theorem 3.1, this (c_*, u_*) belongs to \mathcal{D}_* . In particular $c_* > 0$.

We have shown any branch of stable solutions can be extended for $c \in] - \infty, c_*[$. But by Lemma 2.5 there is a unique stable solution of (1) for $c = 0$, namely $u_\dagger = u_\dagger(a)$. This proves uniqueness. \square

Proof of Lemma 4.3. This lemma is known ([10, Theorem 3.2] and [13, p. 3613]), but for completeness we give the proof. Let (c_*, u_*) be a degenerate solution with Morse index equal to zero. Let t_* and y_* be such that $u_* = t_*w_* + y_*$, with $w_* \in S$ satisfying $\Delta w_* + aw_* - f'(u_*)w_* = 0$, $w_* > 0$, and $y_* \in \mathcal{R}_{w_*}$. We let $\tilde{G} : \mathbb{R} \times \mathcal{R}_{w_*} \times \mathbb{R} \times S \times \mathbb{R} \rightarrow L^p(\Omega) \times L^p(\Omega)$ be defined by

$$\begin{aligned} \tilde{G}(t, y, c, w, \mu) &= (\Delta(tw_* + y) + a(tw_* + y) - f(tw_* + y) - ch, \\ &\quad \Delta w + aw - f'(tw_* + y)w + \mu w). \end{aligned}$$

We have $\tilde{G}(t_*, y_*, c_*, w_*, 0) = 0$. We may use the Implicit Function Theorem to describe the solutions of $\tilde{G} = 0$ in a neighborhood of $(t_*, y_*, c_*, w_*, 0)$. Indeed, at this point,

$$\begin{aligned} \tilde{G}_y z + \tilde{G}_c \gamma + \tilde{G}_w \omega + \tilde{G}_\mu \nu &= (\Delta z + az - f'(t_*w_* + y_*)z - \gamma h, \\ &\quad \Delta \omega + a\omega - f'(t_*w_* + y_*)\omega \\ &\quad - f''(t_*w_* + y_*)zw_* + \nu w_*). \end{aligned}$$

If this derivative vanishes, then we get $\gamma = 0$ and then $z = 0$. This implies $\nu = 0$ and then $\omega = 0$. The derivative is a homeomorphism from $\mathcal{R}_{w_*} \times \mathbb{R} \times \mathcal{R}_{w_*} \times \mathbb{R}$ to $L^p(\Omega) \times L^p(\Omega)$. So the solutions of $\tilde{G} = 0$ in a neighborhood of $(t_*, y_*, c_*, w_*, 0)$ lie on a curve $t \mapsto (t, y(t), c(t), w(t), \mu(t))$. Differentiating $\tilde{G}(t, y(t), c(t), w(t), \mu(t)) = (0, 0)$ once with respect to t ,

$$\Delta z + az - f'(t_*w_* + y_*)z - \gamma h = -(\Delta w_* + aw_* - f'(t_*w_* + y_*)w_*) = 0,$$

$$\Delta \omega + a\omega - f'(t_*w_* + y_*)\omega - f''(t_*w_* + y_*)zw_* + \nu w_* = f''(t_*w_* + y_*)w_*^2$$

Here $z = y'(t_*)$, $\gamma = c'(t_*)$, $\omega = w'(t_*)$ and $\nu = \mu'(t_*)$. Clearly, both γ and z vanish. This implies

$$\nu = \frac{\int f''(t_*w_* + y_*)w_*^3}{\int w_*^2}.$$

It is impossible for $\max_{\Omega}(t_*w_* + y_*) \leq M$ because otherwise $\Delta w_* + aw_* = 0$ with $w_* > 0$. Hence,

$$\mu'(t_*) > 0. \quad (43)$$

Differentiating the first equation in $\tilde{G}(t, y(t), c(t), w(t), \mu(t)) = (0, 0)$ twice with respect to t , at t_* ,

$$\begin{aligned} \Delta z' + az' - f'(t_*w_* + y_*)z' - \gamma'h - f''(t_*w_* + y_*)z(w_* + z) = \\ f''(t_*w_* + y_*)w_*(w_* + z). \end{aligned}$$

Here $z = y'$. This can be rewritten as

$$\begin{aligned} \Delta z' + az' - f'(t_*w_* + y_*)z' - \gamma'h &= f''(t_*w_* + y_*)(w_* + z)^2 \\ &= f''(t_*w_* + y_*)w_*^2, \end{aligned}$$

as $z(t_*) = 0$. Multiplying by w_* and integrating,

$$c''(t_*) = - \frac{\int f''(t_*w_* + y_*)w_*^3}{\int hw_*}.$$

This is formula (2.7) of [13]. So $c''(t_*)$ is negative. We recall from equation (13), a degenerate solution with $a > \lambda_1$ has $c_* > 0$. As t increases from t_* , $c(t)$ decreases and the solution becomes stable. So the “end” of \mathcal{M}^* coincides with the piece of curve parametrized by $t \mapsto (c(t), tw(t) + y(t))$, for t in a right neighborhood of t_* . A parametrization of \mathbf{m}^\sharp is obtained by taking t in a left neighborhood of t_* . \square

Proof of Lemma 4.4. We may apply the Implicit Function Theorem as above as $(c, u) = (0, 0)$ is a nondegenerate solution of (1) when a is not an eigenvalue of the Laplacian. This gives the curve $\check{\mathcal{C}}$ of solutions parametrized by $c \mapsto (c, \check{u}(c))$ for $c \in] - \check{c}, \check{c}[$. To obtain more information about the behavior of the solutions for a close to λ_1 , we look at the linearized equation at $u = 0$,

$$\Delta v + av = h,$$

where $v = \check{u}'(0)$. As in [6, Theorem 2, formula (5)], we write $v = t\phi + y$ with $y \in \mathcal{R}$. This leads to

$$v = \frac{\int h\phi}{a - \lambda_1}\phi + (\Delta + a)^{-1} [h - (\int h\phi)\phi]. \quad (44)$$

Note that as $a \searrow \lambda_1$ the second term of the sum on the right hand side approaches $(\Delta + \lambda_1)^{-1} [h - (\int h\phi)\phi]$. We may choose a right neighborhood of λ_1 small enough so that $v > 0$. From Taylor's formula,

$$\check{u}(c) = 0 + cv + o(c) \quad \text{as } c \rightarrow 0. \quad (45)$$

Reducing the right neighborhood of λ_1 if necessary, we may assume $c_1 < c_2$ implies $\check{u}(c_1) < \check{u}(c_2)$. In particular $\check{u}(c)$ is positive for small c . Note $\check{c} \searrow 0$ as $a \searrow \lambda_1$. \square